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Parameter-free H(div) preconditioning for a mixed finite element formulation of diffusion problems

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Mixed finite element formulations of generalised diffusion problems yield linear systems with ill-conditioned, symmetric and indefinite coefficient matrices. Preconditioners with optimal work complexity that do not rely on artificial parameters are essential. We implement lowest-order Raviart-Thomas elements and analyse practical issues associated with so-called '$H(div)$ preconditioning'. Properties of the exact scheme are discussed in Powell & Silvester (2003), 'Optimal preconditioning for Raviart-Thomas mixed formulation of second-order elliptic problems,' *SIAM J. Matrix Anal. Appl.,* **25**(3), 718–738. We extend the discussion, here, to practical implementation, the components of which are any available multilevel solver for a weighted $H(div)$ operator and a pressure mass matrix. A new bound is established for the eigenvalue spectrum of the preconditioned system matrix and extensive numerical results are presented.

Keywords: saddle point problems, mixed finite elements, Raviart-Thomas, preconditioning.

1. Introduction

Let Ω be a polygon in $\mathbb{R}^d$ $(d = 2,3)$ and consider the boundary value-problem,

$$\begin{aligned} \mathscr{A}^{-1}\vec{u} - \nabla p &= 0, \\ \nabla\cdot\vec{u} &= -f \quad \text{in } \Omega, \\ p &= g \quad \text{on } \partial\Omega_D, \\ \vec{u}\cdot\vec{n} &= 0 \quad \text{on } \partial\Omega_N, \end{aligned} \tag{1..1}$$

where $\partial\Omega_D \neq \emptyset$ and $\mathscr{A} = \mathscr{A}(\vec{x})$ is a $d \times d$ bounded, symmetric and uniformly positive definite matrix-valued function with minimum eigenvalue bounded away from zero.

1.1. *Second level section*

The mixed first-order system (1..1) occurs in models of fluid flow in porous media (see Russell & Wheeler (1983) and Ewing *et al.* (1983).) The macroscopic flow of groundwater satisfies $\vec{u} = -k\mu^{-1}\nabla P_R$, where $\vec{u}$ denotes fluid discharge, P_R is 'residual pressure', k is the permeability coefficient and μ is viscosity. Coupling Darcy's law with mass conservation yields (1..1) with $f = 0$, $\mathscr{A} = -\frac{k}{\mu}\mathscr{I}$ and $p := P_R$. To fix ideas, we call p and $\vec{u} = \mathscr{A}\nabla p$ the 'pressure' and 'velocity' solutions, respectively. Flow domains are often comprised of different media with spatially varying permeability coefficients, leading to heterogeneous problems with discontinuous $\mathscr{A}$. In stratified media, the entries of $\mathscr{A}$ corresponding to different co-ordinate directions vary in magnitude, yielding anisotropic $\mathscr{A}$. Mixed finite element methods are favoured when $\vec{u}$ is the variable of interest as post-processing primal solutions leads to loss of accuracy. Low order mixed methods also conserve mass locally.

The Krylov subspace method MINRES (see Paige & Saunders (1975)) is an optimal solver for (**??**). The kth iterate minimises the Euclidean norm of the kth residual error over the corresponding Krylov space and, since C is sparse, the cost per iteration depends linearly on the problem size. Popular alternative soltution schemes include Uzawa's method and the augmented Lagrangian method (see Fortin & Glowinski (1983, Ch.1)). However, these methods require nested iteration and the user's choice of relaxation parameter determines the convergence of the outer iteration. Deficiencies associated with this are highlighted in Rusten & Winther (1992). Fast convergence is obtained for the outer iteration if the relaxation parameter is tuned in the right way but for the augmented Lagrangian method, that cripples the inner-iteration. Preconditioners for the inner-solve have been suggested, see Hiptmair (1997), but optimal parameter values have not been discussed. Vassilevski & Lazarov (1996) use MINRES but introduce artificial parameters. Applying MINRES directly to (**??**) is simpler and more user-friendly.

LEMMA 1..1 The $(n+m)$ eigenvalues of the generalised eigenvalue problem,

$$\begin{pmatrix} A_I & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \sigma \begin{pmatrix} A_I + D & 0 \\ 0 & N \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix}, \tag{1..2}$$

arising in the Raviart-Thomas approximation of (**??**) are bounded by constants independent of h and lie in the intervals $\left[-1,-\beta_*^2\right] \cup [1]$, where β_* is the discrete inf-sup constant in (**??**).

When $\mathscr{A} = \mathscr{I}$, choosing P_1 and P_2 to represent the norms for which stability holds, leads to an h-optimal eigenvalue bound. For problems with general coefficient tensors, however, P must also supply scaling with respect to $\mathscr{A}$. We choose $P_1 = A + D$, representing the weighted norm $\|\cdot\|_{div,\mathscr{A}}$, induced by the inner-product (**??**).

2. Practical $H(div)$ preconditioning

Now, let V be *any* symmetric and positive definite approximation to $H = A + D$. An ideal choice is a V-cycle of multigrid, the cost of which is known to depend linearly on the problem size (see Trottenberg *et al.* (2001, p.74).) However, *any* available approximation can be substituted, provided that there exist positive constants θ and Θ, satisfying,

$$0 < \theta \leqslant \frac{\underline{u}^T H \underline{u}}{\underline{u}^T V \underline{u}} \leqslant \Theta \leqslant 1 \quad \forall \underline{u} \in I\!R^n \backslash \{\underline{0}\}. \tag{2..1}$$

The condition $\Theta \leqslant 1$ is not restrictive, since the chosen V can always be rescaled. It is purely to simplify presentation in the sequel. In section 5 we perform numerical computations with a particular V and compute corresponding values of θ and Θ.

Now consider the preconditioner,

$$P = \begin{pmatrix} V & 0 \\ 0 & N \end{pmatrix}. \tag{2..2}$$

We require bounds for the eigenvalues of $P^{-1}C$. To simplify notation further, let

$$a = \left(\frac{c\mu_{min}}{|\ T\ |_{min} + \mu_{min}} \right), \tag{2..3}$$

so that the ideal bound (**??**) reads $[-1,-a] \cup [1]$. In the proof, given in Powell & Silvester (2003, Lemma 2.3), we establish that the negative eigenvalues of (**??**) are the values $\{\lambda\}$ satisfying, BH^{-1}

$B^T \underline{p} = -\lambda N \underline{p}$. In other words,

$$0 < a \leqslant \frac{\underline{p}^T B H^{-1} B^T \underline{p}}{\underline{p}^T N \underline{p}} \leqslant 1 \quad \forall \underline{p} \in I\!R^m \backslash \{\underline{0}\}. \tag{2..4}$$

THEOREM 2..1 The $(n+m)$ eigenvalues $\{\lambda_i\}_{i=1}^{n+m}$ of the generalised eigenvalue problem,

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \lambda \begin{pmatrix} V & 0 \\ 0 & N \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix}, \tag{2..5}$$

in the Raviart-Thomas approximation of (**??**), lie in the union of the intervals,

$$\left[-1, \frac{1}{2}\left(\theta(1-a) - \sqrt{\theta^2(a-1)^2 + 4a\theta}\right)\right] \cup [\theta, 1], \tag{2..6}$$

where θ is the constant satisfying (2..1) and a is the constant defined in (2..3).

Proof. First note that since the eigenvalues $\{\lambda_i\}_{i=1}^{m+n}$ satisfy,

$$A\underline{u} + B^T \underline{p} = \lambda V \underline{u}, \qquad B\underline{u} = \lambda N \underline{p},$$

eliminating $\underline{p}$ yields $\lambda \underline{u}^T A \underline{u} + \underline{u}^T D \underline{u} = \lambda^2 \underline{u}^T V \underline{u}$. Applying (2..1) yields,

$$\lambda^2 \underline{u}^T H \underline{u} \leqslant \lambda \underline{u}^T A \underline{u} + \underline{u}^T D \underline{u} \leqslant \lambda^2 \theta^{-1} \underline{u}^T H \underline{u}. \tag{2..7}$$

From the left inequality we obtain,

$$\left(\lambda^2 - \lambda\right) \underline{u}^T A \underline{u} + \left(\lambda^2 - 1\right) \underline{u}^T D \underline{u} \leqslant 0, \tag{2..8}$$

and since A is positive definite and D is semi-positive definite, we immediately establish $|\lambda| \leqslant 1$. The right inequality in (2..7) yields,

$$(1-\lambda)\, \underline{u}^T D \underline{u} \leqslant \left(\lambda^2 \theta^{-1} - \lambda\right) \underline{u}^T H \underline{u}.$$

Since $0 \leqslant 1 - \lambda$ it follows that $0 \leqslant \lambda\left(\lambda\theta^{-1} - 1\right)$. Hence if $\lambda > 0$, we have $\lambda \geqslant \theta$, and the bound for the positive eigenvalues is proved.

Now assume that $\lambda < 0$. Eliminating $\underline{u}$ yields,

$$B\left(\lambda V - A\right)^{-1} B^T \underline{p} = \lambda N \underline{p}. \tag{2..9}$$

The values of λ satisfying (2..9) are the eigenvalues of the matrix,

$$\begin{aligned} N^{-\frac{1}{2}} B\left(\lambda V - A\right)^{-1} B^T N^{-\frac{1}{2}} &= N^{-\frac{1}{2}} B\left(\lambda V - H + D\right)^{-1} B^T N^{-\frac{1}{2}} \\ &= N^{-\frac{1}{2}} B\left(\lambda V - H + B^T N^{-1} B\right)^{-1} B^T N^{-\frac{1}{2}} \\ &= N^{-\frac{1}{2}} B Y^{-\frac{1}{2}} \left(I + Y^{-\frac{1}{2}} B^T N^{-1} B Y^{-\frac{1}{2}}\right)^{-1} Y^{-\frac{1}{2}} B^T N^{-\frac{1}{2}} \\ &= X\left(I + X^T X\right)^{-1} X^T, \end{aligned}$$

where, here, $X = N^{-\frac{1}{2}} B Y^{-\frac{1}{2}}$ and $Y = \lambda V - H$. Applying the Sherman-Morrison-Woodbury formula to $\left(I + XX^T\right)^{-1}$ yields,

$$X\left(I + XX^T\right)^{-1} X^T = X\left(I - X^T\left(I + XX^T\right)^{-1} X\right) X^T. \tag{2..10}$$

Let $\underline{v}$ be an eigenvector of XX^T and let σ denote the corresponding eigenvalue. Then, with (2..10), we obtain,

$$\begin{aligned} X\left(I + XX^T\right)^{-1} X^T \underline{v} &= XX^T \underline{v} - XX^T\left(I + XX^T\right)^{-1} XX^T \underline{v} \\ &= \sigma \underline{v} - \left(\frac{\sigma^2}{1+\sigma}\right)\underline{v} = \left(\frac{\sigma}{1+\sigma}\right)\underline{v}. \end{aligned}$$

Hence, the values $\{\lambda\}$ we are seeking in (2..9) satisfy,

$$\lambda_i = \frac{\sigma_i}{1+\sigma_i}, \tag{2..11}$$

where each σ_i is an eigenvalue of,

$$B\left(\lambda_i V - H\right)^{-1} B^T \underline{p} = \sigma N \underline{p}. \tag{2..12}$$

We now can obtain a bound for these values by exploiting the spectral equivalence of H and V defined by (2..1). (Note that we have no readily available information about the spectral equivalence of the leading blocks A and V.) □

Consider, first, the eigenvalues $\{\mu\}$ of,

$$\left(\lambda V - H\right)^{-1} \underline{u} = \mu H^{-1} \underline{u}. \tag{2..13}$$

Since $(\lambda V - H)$ is negative definite and H^{-1} is positive definite, the values of μ are negative. Rearranging (2..13) and applying (2..1) yields $\theta \leqslant \lambda\mu\left(\mu+1\right)^{-1} \leqslant 1$. Recalling that $\theta > 0$, $\mu < 0$ and $\lambda < 0$, we find that,

$$\mu \in \left[\frac{1}{\lambda - 1}, \frac{\theta}{\lambda - \theta}\right]. \tag{2..14}$$

REMARK 2..1 When $\theta = 1$, we recover the eigenvalue bound (**??**).

EXAMPLE 2..1 Next, introduce a jump in the coefficient and set $\mathscr{A} = \alpha\mathscr{I}$ in one quadrant of Ω so that $\mu_{min} \to 0$ in (2..3) as $\alpha \to 0$ (see Powell & Silvester (2003).) Values of θ are listed in Table **??**; $\Theta = 1$ in all cases. The approximation to $\mathscr{H}_{\mathscr{A}}$ is $\mathscr{A}$-optimal and h-optimal. The negative eigenvalues of the preconditioned saddle point system, for $\alpha = 10^{-3}$ and 10^{-6} are listed in Tables **??**–**??**. Observe that the

TABLE 1 *Eigenvalues of $V^{-1}H$, unit coefficients*

h	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
θ	0.5938	0.4595	0.4273
Θ	1	1	1

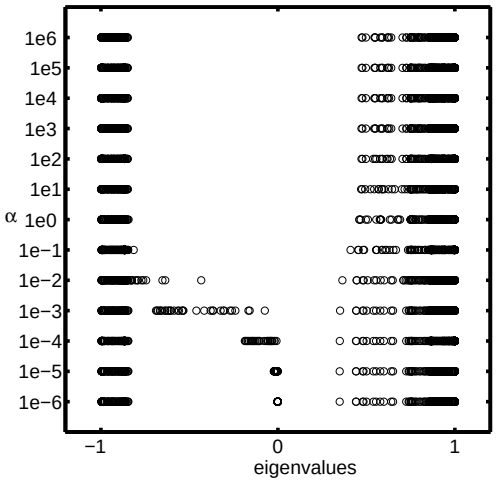

FIG. 1. Eigenvalues of preconditioned saddle point system, $h = \frac{1}{16}, \alpha \in \left[10^{-6}, 10^{6}\right]$

right-hand bound for the negative eigenvalues is tighter as $\alpha \to 0$. The eigenvalues of the preconditioned saddle point system, for fixed h and varying α are plotted in Fig. 1. The scale on the y-axis corresponds to values of $\alpha \in \left[10^{-6}, 10^{6}\right]$ and each line of the plot depicts the eigenvalues for a different value of α.

Clearly, for small values $\alpha << 1$, MINRES convergence will deteriorate. Iteration counts obtained with the ideal preconditioner are listed in columns 2–4 of Table **??**. Counts for the multigrid version are given in columns 5–7. As Theorem 2..1 predicts, the multigrid preconditioner exhibits exactly the same asymptotic behaviour as the ideal version as $\alpha \to 0$.

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TABLE 2 *Theoretical bounds and observed eigenvalues, unit coefficients*

h	*bounds*	*observed*
$\frac{1}{4}$	$[-0.9983, -0.7381] \cup [0.5938, 1]$	$[-0.9879, -0.8507] \cup [0.5943, 1]$
$\frac{1}{8}$	$[-0.9996, -0.6504] \cup [0.4595, 1]$	$[-0.9972, -0.8438] \cup [0.4598, 1]$
$\frac{1}{16}$	$[-0.9999, -0.6503] \cup [0.4273, 1]$	$[-0.9994, -0.8481] \cup [0.4273, 1]$

[a] This is sample notes
[b] This is sample notes

3. Concluding remarks

Motivated by standard stability theory and discussion in Arnold *et al.* (2000) and Powell & Silvester (2003), we described a block-diagonal parameter free preconditioning scheme for the linear systems (**??**) arising in the lowest-order Raviart-Thomas discretisation of generalised diffusion problems. New bounds are established for the eigenvalue spectrum of the preconditioned saddle point matrix when the leading block of the ideal preconditioner is replaced with a suitable approximation. In numerical experiments, we demonstrated the impact of general coefficient tensors on the performance of a particular multilevel approximation and on the theoretical eigenvalue bound.

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