Understanding the Derivative

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1 Introduction

Differentiation is a concept of Mathematics studied in Calculus. There is an ongoing discussion as to who was the first to define differentiation: Leibniz or Newton [1].

Differentiation allows for the calculation of the slope of the tangent of a curve at any given point as shown in Figure 1.

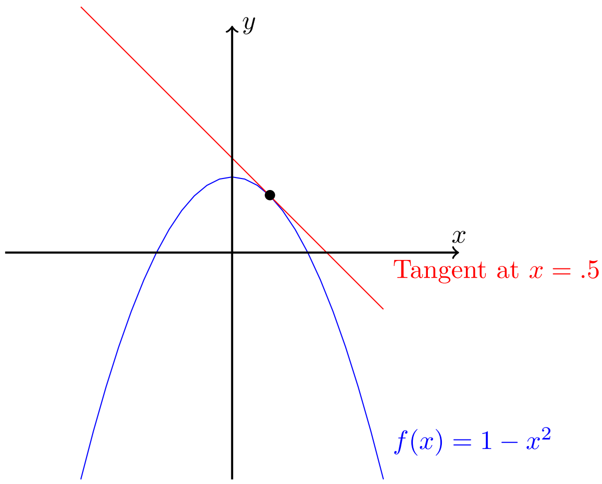

Figure 1: The plot of $f(x) = 1 - x^2$ with a tangent at $x = .5$.

Differentiation is now a technique taught to mathematics students throughout the world. In this document I will discuss some aspects of differentiation.

2 Exploring the derivative using Sage

The definition of the limit of $f(x)$ at $x = a$ denoted as $f'(a)$ is:

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \quad (1)$$

The following code can be used in sage to give the above limit:

```
def illustrate(f, a):
    """
    Function to take a function and illustrate the limiting definition of a derivative at a given po
    """
    lst = []
```

```
for h in srange(.01, 3, .01):
 lst.append([h,(f(a+h)-f(a))/h])
return list_plot(lst, axes_labels=['$x$','$\\frac{f(%.02f+h)-f(%.02f)}{h}$' % (a,a)])
```

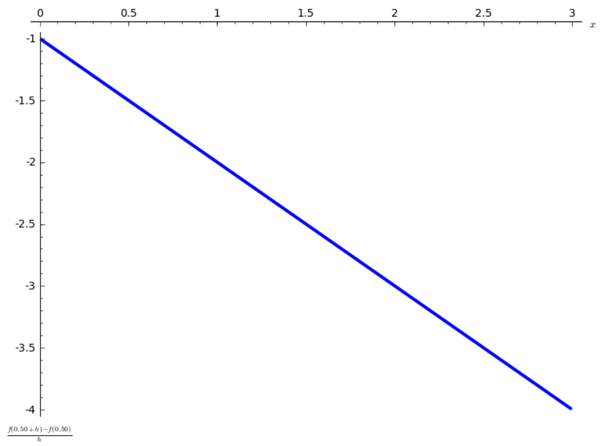

Figure 2: The derivative of $f(x) = 1 - x^2$ at $x = .5$ converging to -1 as $h \to 0$.

If we want to plot the tangent at a point α to a function we can use the following:

$$\begin{aligned} y =& ax + b && \text{(definition of a straight line)} \\ & f'(a)x + b && \text{(definition of the derivative)} \\ & f'(a)x + f(a) - f'(a)a && \text{(we know that the line intersects } f \text{ at } (a, f(a))\text{)} \end{aligned}$$

We can combine this with the approach of the previous piece of code to see how the tangential line converges as the limiting definition of the derivative converges:

```
def convergetangentialline(f, a, x1, x2, nbrofplots=50, epsilon=.1):
    """
    Function to make a tangential line converge
    """
    clrs = rainbow(nbrofplots)
    k = 0
    h = epsilon
    p = plot(f, x, x1, x2)
    while k < nbrofplots:
        tangent(x) = fdash(f, a, h) * x + f(a) - fdash(f, a, h) * a
        p += plot(tangent(x), x, x1, x2, color=clrs[k])
        h += epsilon
        k += 1
    return p
```

The plot shown in Figure 3 shows how the lines shown converge to the actual tangent to $1 - x^2$ as $x = 2$ (the red line is the 'closest' curve).

Note here that the last plot is given using the **real** definition of the derivative and not the approximation.

3 Conclusions

In this report I have explored the limiting definition of the limit showing how as $h \to 0$ we can visualise the derivative of a function. The code involved `https://sage.maths.cf.ac.uk/home/pub/18/` uses the

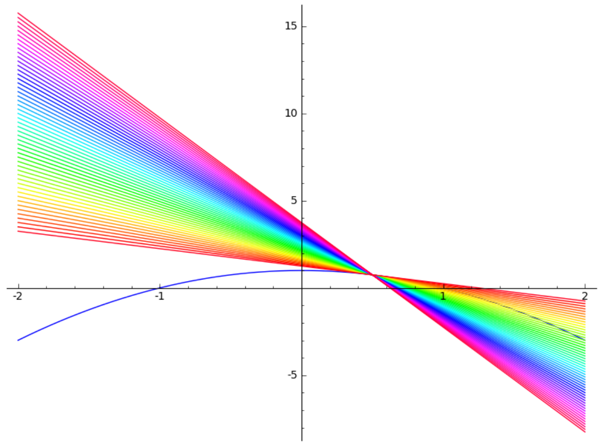

Figure 3: Lines converging to the tangent curve as $h \to 0$.

differentiation capabilities of Sage but also the plotting abilities.

There are various other aspects that could be explored such as symbolic differentiation rules. For example:

$$\frac{dx^n}{dx} = (n+1)x^n \text{ if } x \neq -1$$

Furthermore it is interesting to not that there exists some functions that **are not** differentiable at a point such as the function $f(x) = \sin(1/x)$ which is not differentiable at $x = 0$. A plot of this function is shown in Figure 4.

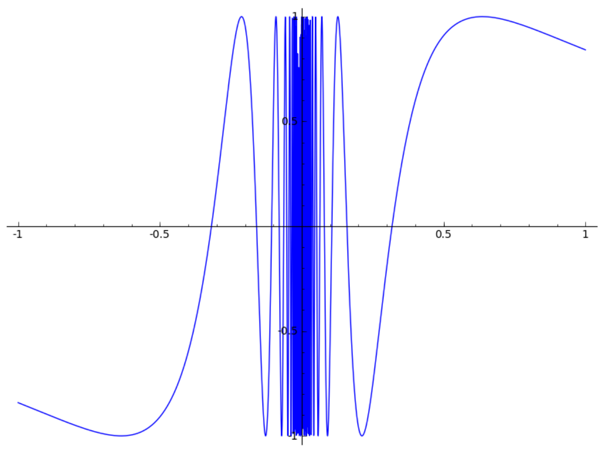

Figure 4: None differentiable function at $x = 0$.

References

[1] Jason Socrates Bardi. The calculus wars: Newton. *Leibniz, and the Greatest Mathematical Clash of All Time (Thunder Mouth, New York, 2006)*, 2006.