The Alexander Polynomial

And all that jazz

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Table of contents

Defining the Polynomial

Assembling The Matrix

- Start with an oriented knot diagram
- Label all the crossings 1, 2, ..., n
- Label all the regions 1, 2, ..., n, n+1, n+2
- Create an n x n+2 matrix where the rows correspond to crossings, the columns correspond to regions.

Assembling The Matrix

Each entry in the matrix will be determined by the labels defined in Figure 1.

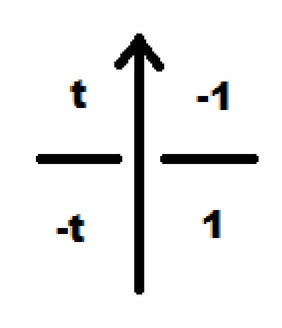

Figure 1: Matrix Entries

At each row, all other regions that do not intersect with the specific crossing have a value of zero.

Assembling The Matrix

- Delete two columns of the matrix corresponding to adjacent regions
- The resultant n x n matrix is the Alexander Matrix
- The determinant of the Alexander Matrix is the Alexander Polynomial
- Depending on which columns are deleted, the determinant may differ by a factor of $\pm t^k$
- Conclude by dividing by the largest possible power of t and factoring out a -1 if necessary to make the constant positive

Example

Trefoil Knot

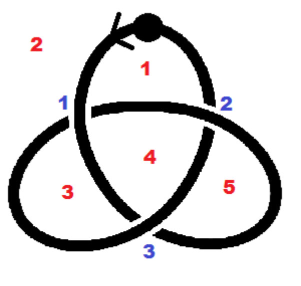

Figure 2: Labeled Trefoil

Corresponding Matrix and Determinant

$$M = \begin{bmatrix} -t & 1 & -1 & t & 0 \\ -1 & 1 & 0 & t & -t \\ 0 & 1 & -t & t & -1 \end{bmatrix}$$

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$$AlexanderMatrix = \begin{bmatrix} -t & 1 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & -t \end{bmatrix}$$

Corresponding Matrix and Determinant

$$M = \begin{bmatrix} -t & 1 & -1 & t & 0 \\ -1 & 1 & 0 & t & -t \\ 0 & 1 & -t & t & -1 \end{bmatrix}$$

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$$\text{Det(Alexander Matrix)} = t^2 - t + 1$$

Reidemeister Moves

Delta-Equivalent Matrices

- Two matrices have the same determinant if they are delta-equivalent and one can be transformed into another by a sequence of the following moves:
- Multiple a row or column by k.
- Swapping two rows or columns.
- Add one row or column to another.
- Add or remove corner.
- Multiply or divide a column by t.

Reidemeister 1 Move

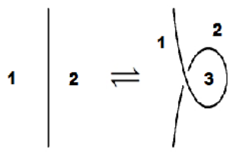

Figure 3: Reidemeister 1 move

- Notice that the R1 move adds one new crossing and one new region
- This corresponds to one new row and one new column in the matrix
- Choose to delete regions 1 and 2 from the matrix
- The new Alexander matrix will have one row and column which only contains only $1, -1, t, or - t$
- Thus the determinant will differ by a factor of -1

Reidemeister 2 Move

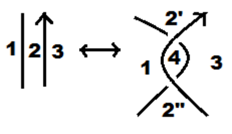

Figure 4: Reidemeister 2 move

- Notice that the R2 move adds 2 new crossings, 1 new region, and splits an existing region into two regions
- Choose to delete region 1 and one of the split regions. We will choose to delete region 2'

Reidemeister 2 Move

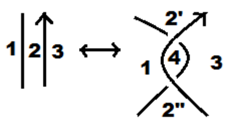

Figure 5: Reidemeister 2 move

- For the given orientation, the entries in the new rows of the matrix are as follows:

$$M = \begin{array}{c} \begin{array}{ccc} 2'' & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & -1 & 1 \\ -t & 1 & -1 \end{pmatrix} \end{array}$$

- The remaining entries in these rows are all zero
- Columns 2” and 3 have nonzero entries below these

Reidemeister 2 Move

- We can add column 4 to column 3

$$M = \begin{array}{c} \begin{array}{ccc} 2'' & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 \\ -t & 0 & -1 \end{pmatrix} \end{array}$$

- Then we can add column 2‘ to column 4

$$M = \begin{array}{c} \begin{array}{ccc} 2'' & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 \\ -t & 0 & 0 \end{pmatrix} \end{array}$$

- Next divide row 2 by -t

$$M = \begin{array}{c} \begin{array}{ccc} 2'' & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \end{array}$$

Reidemeister 2 Move

$$\begin{array}{cc} & \begin{array}{ccc} 2'' & 3 & 4 \end{array} \\ M = & \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \end{array}$$

- When calculating the determinant, expand across the rows with only one nonzero entry
- The determinant will now be unchanged from an R2 move up to a factor of $\pm t^k$

Reidemeister 3 Move

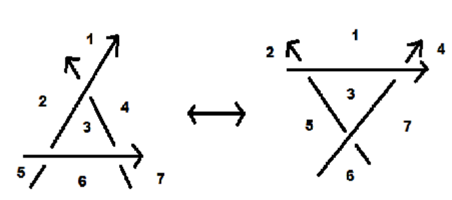

Figure 6: Reidemeister 3 move

- The R3 move changes the matrix dramatically that we fail to identify.
- Notice that two matrices are delta-equivalent if the corresponding matrices with all positive entries are delta-equivalent.

Reidemeister 3 Move

- Due to checkerboard coloring and the way we index the regions around each crossing, we can multiple each odd column by -1 so that each row will have only positive or negative entries.
- We can get a new matrix with non-negative entries by multiplying all negative rows by -1.

Reidemeister 3 Move

- For the given orientation, the entries in the relevant rows of the original matrix are as follows:

$$M = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 0 & -t & t & 0 & 1 & -1 & 0 \\ 0 & 0 & -t & t & 0 & 1 & -1 \\ t & -t & 1 & -1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

- The remaining entries in these rows are all zero
- Only the remaining entries in column 3 are all zero.

Reidemeister 3 Move

- For the given orientation, the entries in the relevant rows of the matrix after R3 move are as follows:

$$N = \begin{array}{c} \begin{array}{ccccccc} 1' & 2' & 3' & 4' & 5' & 6' & 7' \end{array} \\ \begin{pmatrix} -t & 0 & 1 & t & 0 & 0 & -1 \\ t & -t & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & t & 0 & -t & 1 & -1 \end{pmatrix} \end{array}$$

- Again, the remaining entries in these rows are all zero and Only the remaining entries in column 3' are all zero.
- Notice that all entries in other columns remain constant by R3 move.

Reidemeister 3 Move

- Identify whether these two matrices with all positive entries are delta-equivalent.

$$\begin{array}{cc} & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ M' = & \begin{pmatrix} 0 & t & t & 0 & 1 & 1 & 0 \\ 0 & 0 & t & t & 0 & 1 & 1 \\ t & t & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{ccccccc} 1' & 2' & 3' & 4' & 5' & 6' & 7' \end{array} \\ N' = & \begin{pmatrix} t & 0 & 1 & t & 0 & 0 & 1 \\ t & t & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & t & 0 & t & 1 & 1 \end{pmatrix} \end{array}$$

Reidemeister 3 Move

- $$\begin{array}{cc} & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ M' = & \begin{pmatrix} 0 & t & t & 0 & 1 & 1 & 0 \\ 0 & 0 & t & t & 0 & 1 & 1 \\ t & t & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$
- First times row 3 by -t, then add row 2 to row 3, divide column 3 by t, subtract column 3 to column 6, times column 3 by t, add t * row 1 to row 3, add column 3 to column 1, add column 4 to column 2, divide column 3 by t, add -1 * column 2 to column 4, divide column 4 by -1, add -1/t * column 4 to column 5, add -1 * column 4 to column 2, add 1/t * column 1 to column 5, add -1/t * column 4 to column 5, add -1/t * column 2 to column 7, then add 1/t * column 4 to column 7.
- Notice that we transform M' to N' through the sequence of moves stated above.
- The determinant will be unchanged from an R3 move up to a factor of $\pm t^k$

Alexander Polynomial

- If we change our labeling for crossings:
- The regions around each specific crossing remain the same; the row that represents such crossing remains constant.
- Change our labeling for crossings swaps the rows in the polynomial matrix.
- If we change our labeling for regions:
- The crossings that intersect specific region remain the same; the column that represents such region remains constant.
- Change our labeling for regions swaps the columns in the polynomial matrix.

Alexander Polynomial

- Conclusion:
- If the Alexander polynomial for a knot is computed using two different sets of choices for diagrams and labeling, then then two polynomials will differ by a multiple of $\pm t^k$ for some integer k.
- Alexander polynomial is a knot invariant.

Sources

1. JW Alexander, Topological invariants of knots and links, Transactions of the American Mathematical Society, Volume 30, 1928, pp275–306
2. Topological invariants of knots: three routes to the Alexander Polynomial, Edward Long, 2005

Questions?