THE ADDITION FORMULAS FOR THE HYPERBOLIC SINE AND COSINE FUNCTIONS VIA LINEAR ALGEBRA

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ABSTRACT. We present a geometric proof of the addition formulas for the hyperbolic sine and cosine functions, using elementary properties of linear transformations.

1. INTRODUCTION

By analogy with the unit circle, the *unit hyperbola* is the set of points in the plane satisfying the equation $x^2 - y^2 = 1$. The hyperbola is not connected – it has two branches. The right branch ($x > 0$) is parameterized by $x = \cosh t$ and $y = \sinh t$ for $t \in \mathbb{R}$.

A *hyperbolic sector* is the curvilinear triangular region bounded by an arc of the hyperbola and by two line segments from the origin to the endpoints of the arc. If $t > 0$ then the area of the hyperbolic sector bounded by the arc from $(1, 0)$ to $(\cosh t, \sinh t)$ is $t/2$. This fact about hyperbolic sectors provides a *geometric* definition of the hyperbolic sine and cosine functions.

The hyperbolic sine and cosine functions satisfy addition rules that are strikingly similar to the analogous formulas for sine and cosine.

$$\cosh(s + t) = \cosh s \cosh t + \sinh s \sinh t$$
$$\sinh(s + t) = \sinh s \cosh t + \cosh s \sinh t$$

We will prove these formulas under the assumption that s and t are positive, although they are in fact valid for all real values of s and t.

2. PROOF

Let s and t be positive real numbers. The linear transformation

$$T(x, y) = (x \cosh t + y \sinh t, x \sinh t + y \cosh t)$$

preserves the right branch of the unit hyperbola

$$x^2 - y^2 = 1$$

and it preserves areas since $\det T = 1$.

Let A be the hyperbolic sector bounded by the arc from $(1, 0)$ to $(\cosh s, \sinh s)$, and let B be the hyperbolic sector bounded by the arc from $(1, 0)$ to $(\cosh t, \sinh t)$. Note that A has area $s/2$, and B has area $t/2$.

The image $A' := T(A)$ is a hyperbolic sector since T preserves the right branch of the unit hyperbola; and it has area $s/2$ since T preserves areas. A' is bounded by the arc from $T(1, 0) = (\cosh t, \sinh t)$ to

$$T(\cosh s, \sinh s) = (\cosh s \cosh t + \sinh s \sinh t,\ \sinh s \sinh t + \cosh s \cosh t).$$

Now, $A' \cup B$ is a hyperbolic sector, bounded by the arc from $(1, 0)$ to

$$(\cosh s \cosh t + \sinh s \sinh t,\ \cosh s \cosh t + \sinh s \sinh t).$$

Since the area of $A' \cup B$ is $(s + t)/2$, the upper endpoint can be expressed as

$$(\cosh(s + t),\ \sinh(s + t)).$$

Therefore,

$$\cosh(s + t) = \cosh(s) \cosh(t) + \sinh(s) \sinh(t)$$

and

$$\sinh(s + t) = \sinh(s) \sinh(t) + \cosh(s) \cosh(t).$$