SERIES REPRESENTATION OF POWER FUNCTION

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Abstract. In this paper we discuss a problem of generalization of binomial distributed triangle, that is sequence A287326 in OEIS. The main property of A287326 that it returns a perfect cube n as sum of n-th row terms over $k,\ 0 \leq k \leq n-1$ or $1 \leq k \leq n$, by means of its symmetry. In this paper we have derived a similar triangles in order to receive powers $m = 5, 7$ as row items sum and generalized obtained results in order to receive every odd-powered monomial $n^{2m+1},\ m \geq 0$ as sum of row terms of corresponding triangle.
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Contents

1. Structure of the manuscript

The problem of finding expansions of monomials, binomials, trinomials, etc. is classical and a lot of theorems have been found, the most prominent examples are Binomial Theorem [2], Multinomial theorem, Wozpitsky Identity [30], Stirling numbers of second kind identity, etc. In this paper we try to solve the classical problem of finding expansions of monomials. We start from binomial distributed triangle A287326 [11] in OEIS. The main property of A287326 that it returns a perfect cube n as n-th row sum, starting from $0, ..., n-1$ or from $1, ..., n$ by means of its symmetry. Therefore, the following question stated:

- Can we find similar to A287326 triangles in order to receive monomial $n^t,\ t > 3$ as sum of row terms? In other words, can A287326 be generalized in order to receive monomial $n^t,\ t > 3$ as sum of row terms?

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Finding an analogs for $t = 5, 7$ in section 3, we answer to above questions positively. Could this process be continued for each $t = 1, 3, 5, 7...$ similarly? Positive answer to this question is given by theorem (3.29).

2. Introduction

Let describe the derivation of the sequence A287326 in OEIS. Sequence A287326 returns the perfect cube n as row sum over $k,\ 0 \leq k \leq n-1$, as well as sum over $1 \leq k \leq n$, by means of its symmetry. First, consider a difference table of perfect cubes ([4], eq. 7)

(2.1)

n	$\Delta^0(n^3)$	$\Delta^1(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	6
6	216	127	42	6
7	343	169	48	6
8	512	217	54	
9	729	271		
10	1000			

Table 1: Difference table of perfect cubes $n,\ 0 \leq n \leq 10$ up to 3^{rd} order.

Reviewing above table, we have noticed that

$$\begin{aligned}
\Delta(0^3) &= 1 + 6 \cdot 0 = 6\tbinom{1}{2} + \tbinom{1}{0} \\
\Delta(1^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 = 6\tbinom{2}{2} + \tbinom{2}{0} \\
\Delta(2^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 = 6\tbinom{3}{2} + \tbinom{3}{0} \\
\Delta(3^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 = 6\tbinom{4}{2} + \tbinom{4}{0} \\
&\vdots \\
\Delta(n^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + \cdots + 6 \cdot n = 6\tbinom{n+1}{2} + \tbinom{n+1}{0}
\end{aligned} \tag{2.2}$$

Above difference identity is closely related to Faulhaber's sum of cubes, where $n^3 = 6\binom{n+1}{3} + \binom{n+1}{1}$, see ([21], p. 9). Note that $\Delta^2(n^3)$ could be found similarly using above identity $\Delta^2(n^3) = 6\binom{n+1}{3-2} + \binom{n+1}{1-2}$.

Property 2.3. *(Generalized finite difference of power using Faulhaber's formula). Consider the identities, ([21], p. 9).*

$$\begin{cases}
n^1 = & \binom{n}{1} \\
n^3 = & 6\binom{n+1}{3} + \binom{n}{1} \\
n^5 = & 120\binom{n+2}{5} + 30\binom{n+1}{3} + \binom{n}{1}
\end{cases}$$

We can find the first order finite difference of odd power as decreasing the variable of corresponding binomial coefficients by 1, for example

$$\begin{cases}\Delta n^1 = & \binom{n}{0}\\ \Delta n^3 = & 6\binom{n+1}{2}+\binom{n}{0}\\ \Delta n^5 = & 120\binom{n+2}{4}+30\binom{n+1}{2}+\binom{n}{0}\end{cases}$$

Continue similarly, we can express each difference of order $t \geq 1$. The coefficients $\{1, 6, 1, 120, 30, 1\}$ in above identities are generated by

$$V_{n,k} = \frac{1}{r}\sum_{j=0}^{r}(-1)^j\binom{2r}{j}(r-j)^{2n}, \tag{2.4}$$

where $r = n-k+1$, this formula was provided by Peter Luschny in [27]*. Therefore, for every odd $t > 0$ and $m \geq 0$, we have*

$$\Delta^t n^{2m+1} = \sum_{\substack{0\leq k\leq m\\ l\leq 2(m-k)+1-t\\ l \text{ is even}}} V_{m,k}\binom{n+m-k}{l}, \text{ if } t>0 \text{ and odd}$$

Let be $m \geq 0$, $t > 1$ and even, then

$$\Delta^t n^{2m+1} = \sum_{\substack{0\leq k\leq m\\ l\leq 2(m-k)+1-t\\ l \text{ is odd}}} V_{m,k}\binom{n+m-k}{l}, \text{ if } t>1 \text{ and even}$$

Let show finite differences, set $m \geq 1$, $t > 1$, then we have finite difference identity

$$\Delta^t n^{2m} = \sum_{\substack{0\leq k\leq m\\ l\leq 2(m-k)+1-t\\ l \text{ is even}}} \frac{1}{n}V_{m,k}\binom{n+m-k}{l}, \text{ if } t>0 \text{ and odd}$$

And

$$\Delta^t n^{2m} = \sum_{\substack{0\leq k\leq m\\ l\leq 2(m-k)+1-t\\ l \text{ is odd}}} \frac{1}{n}V_{m,k}\binom{n+m-k}{l}, \text{ if } t>1 \text{ and even}$$

By the identity $\sum_{k=0}^{n-1}\Delta n^m = n^m$, we have right to represent perfect cube n as

$$n^3 = 6\tbinom{1}{2}+\tbinom{1}{0}+6\tbinom{2}{2}+\tbinom{2}{0}+6\tbinom{3}{2}+\tbinom{3}{0}+\cdots+6\tbinom{n+1}{2}+\tbinom{n+1}{0} \tag{2.5}$$

Let rewrite it again and display every binomial coefficient as summation $\binom{n+1}{2} = 1+2+\cdots+n$, then

$$n^3 = (1+6\cdot 0)+(1+6\cdot 0+6\cdot 1)+\cdots+(1+6\cdot 0+\cdots+6\cdot(n-1))$$

Particularizing above expression, we get

$$n^3 = n+(n-0)\cdot 6\cdot 0+(n-1)\cdot 6\cdot 1+\cdots+(n-(n-1))\cdot 6\cdot(n-1) \tag{2.6}$$

Provided that n is natural. Now we apply a compact sigma notation on (2.6), thus

$$n^3 = n+\sum_{1\leq k\leq n} 6k(n-k) \tag{2.7}$$

As sum $\sum_{1\leq k\leq n} 6k(n-k)$ consists of n terms, we have right to move n in (2.7) under sigma notation, we get

$$n^3 = \sum_{1\leq k\leq n} 6k(n-k)+1 \tag{2.8}$$

Property 2.9. *(Proof of symmetry). Let be a sets* $A(n) := \{1,\ 2,\ \ldots,\ n\}$, $B(n) := \{0,\ 1,\ \ldots,\ n\}$, $C(n) := \{0,\ 1,\ \ldots,\ n-1\}$, *let be expression (2.8) defined as*

$$M(n,\ C(n)) \overset{\text{def}}{=} \sum_{k\in C(n)} 6k(n-k)+1$$

where x is natural-valued variable and $C(n)$ is iteration set of (2.8), then we have equality

$$M(n,\ A(n)) = M(n,\ C(n)) \tag{2.10}$$

Let review and define expression (2.6) as

$$U(n,\ C(n)) \overset{\text{def}}{=} n + 6\cdot\sum_{k\in C(n)} k(n-k)$$

then

$$U(n,\ A(n)) = U(n,\ B(n)) = U(n,\ C(n)) \tag{2.11}$$

Other words, changing of iteration sets of(2.6) and (2.8) by $A(n),\ B(n),\ C(n)$ and $A(n),\ C(n)$, respectively, doesn't change resulting value for each natural x.

Proof. Let be a plot $y(n,k) = 6k(n-k)+1,\ k\in\mathbb{R},\ 0\leq k\leq 10$, given $n=10$

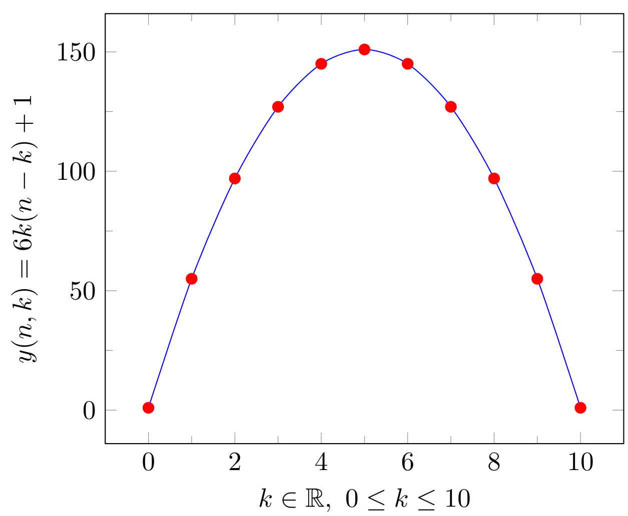

Figure 2. Plot of $6k(n-k)+1,\ k\in\mathbb{R},\ 0\leq k\leq n$, where $n=10$.

Obviously, being a parabolic function, it's symmetrical over $\frac{n}{2}$, hence equivalent $M(n,\ A(n)) = M(n,\ C(n))$ follows. Reviewing (2.6) and denote $u(n,k) = kn - k^2$, we can conclude, that $u(n,0) = u(n,n) = 0$, then equality of $U(n,\ A(n)) = U(n,\ B(n)) = U(n,\ C(n))$ immediately follows. This completes the proof. □

Review above property (2.9). Let be an example of triangle built using

Definition 2.12. For every $n \geq 0$

$$L_1(n,k) \stackrel{\text{def}}{=} 6k(n-k)+1,\ 0 \leq k \leq n \tag{2.13}$$

over n from 0 to $n = 4$, where n denotes corresponding row and k shows the item of row n.

(2.14)

Row 0:					1					
Row 1:				1		1				
Row 2:			1		7		1			
Row 3:		1		13		13		1		
Row 4:	1		19		25		19		1	

Figure 3. Triangle generated by $L_1(n,k)$ from 0 to $n = 4$, sequence A287326 in OEIS, [11].

Note that n-th row sum of Triangle (2.14) over $0 \leq k \leq n-1$ returns perfect cube n. We can see that each row with respect to variable $n = 0,\ 1,\ 2,\ 3,\ 4, ...,$ has Binomial distribution of row terms. One could compare Triangle (2.14) with Pascal's triangle [1], [12]

Row 0:					1				
Row 1:				1		1			
Row 2:			1		2		1		
Row 3:		1		3		3		1	
Row 4:	1		4		6		4		1

Figure 4. Pascal's triangle read by rows, sequence A007318 in OEIS, [1].

Let us approach to show a few properties of triangle (2.14) and $L_1(n,k)$.

Properties 2.15. *Properties of triangle (2.14).*

(1) *Summation of items $L_1(n,k)$ of n-th row of triangle (2.14) over k from 0 to $n-1$ returns perfect cube $n \geq 0$ as follows*

$$\sum_{1 \leq k \leq n} L_1(n,k) = n^3 \tag{2.16}$$

(2) *Relation between $\alpha_{0,n}$ and $\alpha_{1,n}$*

$$\alpha_{0,n+1} = \alpha_{1,n},\ n \geq 1$$

(3) *First item of each row's number corresponding to central polygonal numbers sequence $a(n) = \frac{n^2+n+2}{2}$ (sequence A000124 in OEIS, [13]) returns finite difference of consequent perfect cubes. For example, let be a k-th row of triangle (2.14), such that $k = \frac{n^2+n+2}{2},\ n = 0,1,2,...,$ then item*

$$L_1\left(\frac{n^2+n+2}{2}, 1\right) = (n+1)^3 - n^3 \tag{2.17}$$

(4) *Items of (2.14) have Binomial distribution over rows.*

(5) *Linear recurrence, for every k and $n > 0$*

$$2L_1(n,k) = L_1(n+1,k) + L_1(n-1,k) \tag{2.18}$$

This linear recurrence is direct result of second order binomial transform of $L_1(n,k)$ over n.

(6) *Linear recurrence, for each $n > k$*

$$2L_1(n,k) = L_1(2n-k,k) + L_1(2n-k,0) \tag{2.19}$$

(7) *From (1.24) for every $n \geq 0$ follows*

$$\sum_{1\leq k\leq n} L_1(n,k) = \sum_{1\leq k\leq n} L_1\left(\frac{n^2+n+2}{2}, 1\right) = n^3 \tag{2.20}$$

(8) *Triangle (2.14) is symmetric, i.e*

$$L_1(n,k) = L_1(n,n-k) \tag{2.21}$$

Property 2.22. *(Generalized binomial series by means of identity (2.16). Let review identity (2.16) in sense of*

$$\sum_{1\leq k\leq t} L_1(n,k) = \alpha_{0,t}n - \beta_{0,t}$$

By property (2.9) we rewrite above expression as

$$\sum_{0\leq k\leq t} L_1(n,k) = \alpha_{1,t}n - \beta_{1,t}$$

where subscripts $0,t$ and $1,t$ denote the ranges of summation, respectively. Running over $t > 0$ above identities produce sets of coefficients $\{\alpha_{0,t}\}_t$, $\{\beta_{0,t}\}_t$, $\{\alpha_{1,t}\}_t$ and $\{\beta_{1,t}\}_t$. Below table shows initial terms of these sequences

t	$\alpha_{0,t}$	$\beta_{0,t}$	$\alpha_{1,t}$	$\beta_{1,t}$
1	1	0	6	5
2	6	4	18	28
3	18	27	36	81
4	36	80	60	176
5	60	175	90	325
6	90	324	126	540
7	126	539	168	833
8	168	832	216	1216
9	216	1215	270	1701
10	270	1700	330	2300

Table 5. Array of coefficients $\alpha_{\overline{0,1},n}$, $\beta_{\overline{0,1},n}$ given $n = 1, ..., 10$.

Therefore, perfect cube n could be rewritten as binomials of the form

$$n^3 = \begin{cases} \alpha_{0,n-1}n - \beta_{0,n-1}, & \text{if } t = n-1; \\ \alpha_{1,n}n - \beta_{1,n}, & \text{if } t = n \end{cases}$$

By the main power property, for every $m \in \mathbb{N}$

$$n^m = \begin{cases} \alpha_{0,n-1}n^{m-2} - \beta_{0,n-1}n^{m-3} \\ \alpha_{1,n}n^{m-2} - \beta_{1,n}n^{m-3} \end{cases}$$

We denote above equation as

$$n^m = \alpha_{\overline{0,1},\overline{n-1,n}} n^{m-2} - \beta_{\overline{0,1},\overline{n-1,n}} n^{m-3}$$

Let rewrite the right part of above expression regarding to itself as recursion

$$\begin{aligned} n^m &= \alpha_{\overline{0,1},\overline{n-1,n}}(\alpha_{\overline{0,1},\overline{n-1,n}} n^{m-4} - \beta_{\overline{0,1},\overline{n-1,n}} n^{m-5}) \\ &- \beta_{\overline{0,1},\overline{n-1,n}}(\alpha_{\overline{0,1},\overline{n-1,n}} n^{m-5} - \beta_{\overline{0,1},\overline{n-1,n}} n^{m-6}) \\ &= \alpha^2_{\overline{0,1},\overline{n-1,n}} n^{m-4} - 2\alpha_{\overline{0,1},\overline{n-1,n}}\beta_{\overline{0,1},\overline{n-1,n}} n^{m-5} + \beta^2_{\overline{0,1},\overline{n-1,n}} n^{m-6} \end{aligned}$$

We can observe corresponding binomial coefficient present before each $\alpha_{\overline{0,1},\overline{n-1,n}}$ *times* $\beta_{\overline{0,1},\overline{n-1,n}}$. *Continuous* j*-times recursion gives*

$$n^m = \sum_{k\geq 0}^{\infty} (-1)^k \binom{j}{k} \alpha^{j-k}_{\overline{0,1},\overline{n-1,n}} \beta_{\overline{0,1},\overline{n-1,n}} n^{m-2j-k}, \ j \geq 0$$

Sequences $\alpha_{1,t}, \ \alpha_{0,t>1}$ *are generated by* $3n^2+3n$, *sequence A028896 in OEIS,* [23]. *Sequence* $\beta_{1,t}$ *is generated by* $2n^3+3n^2$, *sequence A275709 in OEIS,* [20].

In this section we have reached binomial distributed triangle (2.14), such that perfect cube n could be found as sum of n-th row terms of (2.14). Therefore, the follow question is stated

Question 2.23. *Can we find similar to A287326 triangles in order to receive monomial* $n^t, \ t > 3$ *as sum of row terms? Is it exist* $L_v(n,k), \ v \neq 1$, *such that*

$$n^t \ \equiv \sum_{1\leq k\leq n} L_v(n,k), \ v \neq t \ ?$$

3. Generalization of sequence A287326

In order to get analogs of Triangle (2.14) one should solve a system of equations, where unknowns are coefficients of polynomial and variable of polynomial is $k(n-k)$. Let show a triangle generated by $L_2(n,k)$, such that sum of n-th row terms returns n^5.

Example 3.1. We suspect that n-th row of triangle is generated by

$$L_2(n,k) = A_{2,2}(n-k)^2k^2 + A_{2,1}(n-k)k + A_{2,0} \tag{3.2}$$

where $A_{2,2}, A_{2,1}, A_{2,0}$ are unknown coefficients and $n \geq 0, \ 0 \leq k \leq n$. Assume that for every $n \geq 0, \ m \geq 0$ holds

$$\sum_{1\leq k\leq n} L_2(n,k) \ \equiv \ n^5 \tag{3.3}$$

In more explicit view

$$
\begin{aligned}
(3.4)\qquad & A_{2,2}\sum_{1\le k\le n} k^2(n-k)^2 + A_{2,1}\sum_{1\le k\le n} k(n-k) + A_{2,0}n \\
= \quad & A_{2,2}\sum_{1\le k\le n} k^2(n^2-2nk+k^2) + A_{2,1}\sum_{1\le k\le n} kn - k^2 + A_{2,0}n \\
= \quad & A_{2,2}\sum_{1\le k\le n} k^2n^2 - 2nk^3 + k^4 + A_{2,1}\sum_{1\le k\le n} kn - k^2 + A_{2,0}n \\
= \quad & A_{2,2}n^2\sum_{1\le k\le n} k^2 - 2A_{2,2}n\sum_{1\le k\le n} k^3 + A_{2,2}\sum_{1\le k\le n} k^4 + A_{2,1}n\sum_{1\le k\le n} k \\
- \quad & A_{2,1}\sum_{1\le k\le n} k^2 + A_{2,0}n
\end{aligned}
$$

Thus, we have received expression containing sums of powers of successive natural numbers, where powers are $\{1,2,3,4\}$. By the Faulhaber's formula [7], the following identities hold

$$\sum_{1\le k\le n} k = \frac{n^2+n}{2}, \tag{3.5}$$

$$\sum_{1\le k\le n} k^2 = \frac{2n^3+3n^2+n}{6}, \tag{3.6}$$

$$\sum_{1\le k\le n} k^3 = \frac{n^4+2n^3+n^2}{4}, \tag{3.7}$$

$$\sum_{1\le k\le n} k^4 = \frac{6n^5+15n^4+10n^3-n}{30}. \tag{3.8}$$

Now we substitute above identities to (3.4), respectively, we get

$$
\begin{aligned}
& A_{2,2}n^2\frac{2n^3+3n^2+n}{6} - 2A_{2,2}n\frac{n^4+2n^3+n^2}{4} + A_{2,2}\frac{6n^5+15n^4+10n^3-n}{30} \\
+ \quad & A_{2,1}n\frac{n^2+n}{2} - A_{2,1}\frac{2n^3+3n^2+n}{6} + A_{2,0}n
\end{aligned}
$$

Particularizing the elements of above expression and moving them under the common divisor, we get

$$\frac{A_{2,2}n^5 - A_{2,2}n + 30A_{2,0}}{30} + A_{2,1}\left(\frac{n^3-n}{6}\right) \tag{3.9}$$

We have to remember that expression (3.9) is the left side of the input equation (2.2). Therefore,

$$\frac{A_{2,2}n^5 - A_{2,2}n + 30A_{2,0}}{30} + A_{2,1}\left(\frac{n^3-n}{6}\right) = n^5,\ n \ge 0 \tag{3.10}$$

In order to satisfy (3.10) for each natural n, coefficients $A_{2,0}, A_{2,1}, A_{2,2}$ should be a solutions of following system of equations

$$
\begin{cases}
\frac{1}{30}A_{2,2} & = 1 \\
A_{2,1} & = 1 \\
30A_{2,0} - A_{2,2} & = 0
\end{cases}
$$

The only solution of above system is $A_{2,2} = 30,\ A_{2,1} = 0,\ A_{2,0} = 1$. Hereby, $L_2(n,k)$ takes the form

$$L_2(n,k) = 30k^2(n-k)^2 + 1 \tag{3.11}$$

And for each natural n holds

$$\sum_{1\le k\le n} 30k^2(n-k)^2 + 1 \ = \ n^5 \tag{3.12}$$

Let show initial rows of triangle built by $L_2(n,k)$

$$\begin{array}{ccccccccccc} &&&&& 1 &&&&& \\ &&&& 1 && 1 &&&& \\ &&& 1 && 31 && 1 &&& \\ && 1 && 121 && 121 && 1 && \\ & 1 && 271 && 481 && 271 && 1 & \\ 1 && 480 && 1081 && 1081 && 481 && 1 \\ \dots &&&&&&&&&& \end{array} \tag{3.13}$$

Figure 6. Triangle generated by $L_2(n,k),\ 0 \le k \le n$, sequence A300656 in OEIS, [15].

Similarly, finding the coefficients $A_{3,0}, A_{3,1}, A_{3,2}, A_{3,3}$ in

$$L_3(n,k) = A_{3,3}k^3(n-k)^3 + A_{3,2}k^2(n-k)^2 + A_{3,1}k(n-k) + A_{3,0} \tag{3.14}$$

we get $A_{3,3} = 140,\ A_{3,2} = -14,\ A_{3,1} = 0,\ A_{3,0} = 1$, therefore, for each $n \ge 0$ holds

$$\sum_{1\le k\le n} 140k^3(n-k)^3 - 14k^2(n-k)^2 + 1 \ = \ n^7 \tag{3.15}$$

Below we show a few initial rows of triangle built by $L_3(n,k)$

$$\begin{array}{ccccccccccc} &&&&& 1 &&&&& \\ &&&& 1 && 1 &&&& \\ &&& 1 && 127 && 1 &&& \\ && 1 && 1093 && 1093 && 1 && \\ & 1 && 3793 && 8905 && 3793 && 1 & \\ 1 && 8905 && 30157 && 30157 && 8905 && 1 \\ \dots &&&&&&&&&& \end{array} \tag{3.16}$$

Figure 7. Triangle generated by $L_3(n,k),\ 0 \le k \le n$, sequence A300785 in OEIS, [16].

We assume now that generalization of A287326 holds for odd powers only. To generalize our sequences A287326, A300656, A300785 for every odd power $2m+1,\ m = 0, 1, 2...$ we have to review the generating functions of corresponding sequences, that is

$$\sum_{1\le k\le n} \sum_{0\le j\le m} A_{m,j}k^j(n-k)^j = n^{2m+1},\ m = 1,2,3 \tag{3.17}$$

Where $A_{m,j}$ are unknown coefficients of polynomials (2.1) and (2.13).

Definition 3.18. Let define the part of (2.1) as

$$\sum_{0\le j\le m} A_{m,j}k^j(n-k)^j \stackrel{\text{def}}{=} L_m(n,k) \stackrel{\text{def}}{=} \sum_{0\le j\le m} A_{m,j}T^j(n,k)$$

where

$$T(n,k) \stackrel{\text{def}}{=} k(n-k).$$

Note that $L_m(n,k)$ is generalization of definitions (2.12) for $m=1$ and (3.11) for $m=2$, respectively.

For example, generating functions of sequences A287326, A300656, A300785 are

$$\begin{cases} L_1(n,k) = 1 + 6k(n-k), & \text{for } A287326 \\ L_2(n,k) = 1 - 0k(n-k) + 30k^2(n-k)^2, & \text{for } A300656 \\ L_3(n,k) = 1 - 14k(n-k) - 0k^2(n-k)^2 + 140k^3(n-k)^3, & \text{for } A300785 \end{cases}$$

Where coefficients $A_{m,j}$, for $m = 1,2,3$ are $\{A_{1,j}\}_{j=0}^{1} = \{1,6\}$, $\{A_{2,j}\}_{j=0}^{2} = \{1,0,30\}$, $\{A_{3,j}\}_{j=0}^{3} = \{1,-14,0,140\}$ in definitions of generating functions of A287326, A300656, A300785, respectively. To generalize above result in order to receive monomial n^{2m+1} as $\sum_{1\le k\le n} L_m(n,k) = n^{2m+1},\ m = 0,1,2,...$ one has to solve the system of equations. Complete set of coefficients $\{A_{m,0},\ \dots,\ A_{m,m}\}$ such that $\sum_{1\le k\le n} L_m(n,k) = n^{2m+1},\ m \ge 0$ holds can be found solving follow system of equations

$$\begin{cases} L_m(1,0) = 1^{2m+1} \\ L_m(2,0) + L_m(2,1) = 2^{2m+1} \\ L_m(3,0) + L_m(3,1) + L_m(3,2) = 3^{2m+1} \\ \vdots \\ L_m(r,0) + L_m(r,1) + \cdots + L_m(r,r-1) = r^{2m+1},\ r \ge m \end{cases} \tag{3.19}$$

List of solutions[1] of system (2.4) is split and assigned to OEIS under the numbers A302971 (numerators of $A_{m,j}$) and A304042 (denominators of $A_{m,j}$). To reach recurrent formula of $A_{m,j}$, first let fix the unused values $A_{m,j} = 0,$ for $j < 0$ or $j > m$, so we don't need to care about the summation range for j, then by expanding $(n-k)^j$ and using Faulhaber's formula [7], we get

$$\begin{aligned} \sum_{k=0}^{n-1}(n-k)^j k^j &= \sum_{k=0}^{n-1}\sum_{i}^{\infty}\binom{j}{i} n^{j-i}(-1)^i k^{i+j} \qquad (3.20) \\ &= \sum_{i}^{\infty}\binom{j}{i} n^{j-i}\frac{(-1)^i}{i+j+1}\left[\sum_{t}^{\infty}\binom{i+j+1}{t}B_t n^{i+j+1-t} - B_{i+j+1}\right] \\ &= \underbrace{\sum_{i,t}^{\infty}\binom{j}{i}\frac{(-1)^i}{i+j+1}\binom{i+j+1}{t}B_t n^{2j+1-t}}_{(\star)} - \underbrace{\sum_{i}^{\infty}\binom{j}{i}\frac{(-1)^i}{i+j+1}B_{i+j+1}n^{j-i}}_{(\diamond)} \end{aligned}$$

---

[1] One can produce a list of solutions of system (2.4) up to $t = 11$ using Mathematica code solutions_system_2_4.txt, [24].

where B_t are Bernoulli numbers [14]. Now, we notice that

$$\sum_{i}^{\infty}\binom{j}{i}\frac{(-1)^i}{i+j+1}\binom{i+j+1}{t}=\begin{cases}\frac{1}{(2j+1)\binom{2j}{j}}, & \text{if } t=0;\\ \frac{(-1)^j}{t}\binom{j}{2j-t+1}, & \text{if } t>0\end{cases} \tag{3.21}$$

In particular, the last sum is zero for $0<t\le j$. Now we substitute the terms from right part of (3.25) into ($\star$), thus

$$\sum_{i,t}^{\infty}\binom{j}{i}\frac{(-1)^i}{i+j+1}\binom{i+j+1}{t}B_t n^{2j+1-t} = \frac{1}{(2j+1)\binom{2j}{j}} + \sum_{t>0}\frac{(-1)^j}{t}\binom{j}{2j-t+1}B_t n^{2j+1-t}$$

Therefore, (3.24) takes the form

$$(*)\quad \sum_{k=0}^{n-1}(n-k)^j k^j = \underbrace{\frac{1}{(2j+1)\binom{2j}{j}}+\sum_{t>0}\frac{(-1)^j}{t}\binom{j}{2j-t+1}B_t n^{2j+1-t}}_{(\star)} - \underbrace{\sum_{i}^{\infty}\binom{j}{i}\frac{(-1)^i}{i+j+1}B_{i+j+1}n^{j-i}}_{(\diamond)}$$

Now, we keep our attention to $(*)$ and we have to remember that if the sum over some variable i contains $\binom{j}{i}$, then instead of limiting its summation range to $i=0,...,j$, we can let $i=-\infty,...,+\infty$ since $\binom{j}{i}=0$ for i outside the range $i=0,...,j$ (i.e., when $i<0$ or $i>j$). It's much easier to review such sum as summing from $-\infty$ to $+\infty$ (unless specified otherwise), where only a finite number of terms are nonzero, this fact is discussed in [28] as well. To combine or cancel identical terms across the two sums in $(*)$ more easily, we introduce $\ell=2j+1-t$ to ($\star$) and $\ell=j-i$ to ($\diamond$), we get

$$\begin{aligned}\sum_{k=0}^{n-1}(n-k)^j k^j &= \frac{1}{(2j+1)\binom{2j}{j}}n^{2j+1}+\sum_{\ell=-\infty}^{\infty}\frac{(-1)^j}{2j+1-\ell}\binom{j}{\ell}B_{2j+1-\ell}n^{\ell}\\ &- \sum_{\ell=-\infty}^{\infty}\binom{j}{\ell}\frac{(-1)^{j-\ell}}{2j+1-\ell}B_{2j+1-\ell}n^{\ell}\\ &= \frac{1}{(2j+1)\binom{2j}{j}}n^{2j+1}+2\sum_{\text{odd } \ell}^{\infty}\frac{(-1)^j}{2j+1-\ell}\binom{j}{\ell}B_{2j+1-\ell}n^{\ell}.\end{aligned} \tag{3.22}$$

Now, using the definition of $A_{m,j}$, we obtain the following identity for polynomials in n

$$\sum_{j}^{\infty}A_{m,j}\frac{1}{(2j+1)\binom{2j}{j}}n^{2j+1}+2\sum_{j,\ \text{odd } \ell}^{\infty}A_{m,j}\binom{j}{\ell}\frac{(-1)^j}{2j+1-\ell}B_{2j+1-\ell}n^{\ell} \equiv n^{2m+1}. \tag{3.23}$$

Taking the coefficient of n^{2m+1} in above expression, we get $A_{m,m}=(2m+1)\binom{2m}{m}$, and taking the coefficient of x^{2d+1} for an integer d in the range $m/2\le d<m$ we

get $A_{m,d}=0$. Taking the coefficient of n^{2d+1} in (2.8) for $m/4 \le d < m/2$, we get

$$A_{m,d}\frac{1}{(2d+1)\binom{2d}{d}}+2(2m+1)\binom{2m}{m}\binom{m}{2d+1}\frac{(-1)^m}{2m-2d}B_{2m-2d}=0, \tag{3.24}$$

i.e

$$A_{m,d}=(-1)^{m-1}\frac{(2m+1)!}{d!d!m!(m-2d-1)!}\frac{1}{m-d}B_{2m-2d}. \tag{3.25}$$

Continue similarly, we can express $A_{m,j}$ for each integer j in range $m/2^{s+1} \le j < m/2^s$ (iterating consecutively $s=1,2,...$) via previously determined values of $A_{m,d},\ d<j$ as follows

$$A_{m,j}=(2j+1)\binom{2j}{j}\sum_{d=2j+1}^{m}A_{m,d}\binom{d}{2j+1}\frac{(-1)^{d-1}}{d-j}B_{2d-2j}. \tag{3.26}$$

The same formula holds also for $m=0$. Note that in above sum m have to be $m \ge 2j+1$ to return nonzero term $A_{m,j}$.

Definition 3.27. We define here a generalized sequence of coefficients $A_{m,j}$, such that $\sum_{k=0}^{n-1}\sum_{j=0}^{m}A_{m,j}(n-k)^jk^j=n^{2m+1},\ n\ge 0,\ m=0,1,2,...$

$$A_{m,j}:=\begin{cases}0, & \text{if } j<0 \text{ or } j>m\\ (2j+1)\binom{2j}{j}\sum_{d=2j+1}^{m}A_{m,d}\binom{d}{2j+1}\frac{(-1)^{d-1}}{d-j}B_{2d-2j}, & \text{if } 0\le j<m\\ (2j+1)\binom{2j}{j}, & \text{if } j=m\end{cases}$$

Five initial rows of triangle generated by $A_{m,j}$ are

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 6 & & & & \\ & & & 1 & & 0 & & 30 & & & \\ & & 1 & & -14 & & 0 & & 140 & & \\ & 1 & & -120 & & 0 & & 0 & & 630 & \\ 1 & & -1386 & & 660 & & 0 & & 0 & & 2772 \\ \dots & & & & & & & & & & \end{array} \tag{3.28}$$

Figure 8. Triangle generated by $A_{m,j},\ 0\le j\le m$, sequences A302971 (numerators of $A_{m,j}$) and A304042 (denominators of $A_{m,j}$).

Note that starting from row $m \ge 11$ the terms of Triangle (3.28) consist fractional numbers, for example, $A_{11,1}=800361655623,6$. One can find complete list of the numerators and denominators of $A_{m,j}$ in OEIS under the identifiers A302971 and A304042, respectively, see [17],[18]. To verify the terms that definition (3.27) produces one should refer to Mathematica code[2]. Hereby, let be theorem

Theorem 3.29. *For every positive integers n and m holds*

$$\sum_{1\le k\le n}\sum_{j}A_{m,j}k^j(n-k)^j=n^{2m+1}$$

[2] def_2_12.txt, [25]

One can verify results concerning above theorem via Mathematica code[3]. Therefore, theorem (3.29) answers to the question question (2.23) positively, since for every $m \geq 0$ exists a triangle, generated by $\sum_j A_{m,j} k^j (n-k)^j = n^{2m+1}$, such that odd power n^{2m+1} can be reached as sum of n-th row of corresponding triangle over k and A287326 is partial case for $m = 1$.

3.1. **Properties of $L_m(n,k)$ and $A_{m,j}$.** Here we show a few properties of definition $L_m(n,k)$, some of them correlates with properties of partial case $L_1(n,k)$ in 2.15.

(1) Sum of $A_{m,j},\ m \geq 0$ gives

$$\sum_{j \geq 0} A_{m,j} \ = \ 2^{2m+1} - 1$$

(2) Similarly to particular property (1.28), items of $\{L_m(n,k)\}_{k=0}^{n},\ m \geq 0$ is symmetric, i.e

$$L_m(n,k) = L_m(n, n-k),\ n \geq 0,\ 0 \leq k \leq n$$

(3) From (2) for every $n \geq 0,\ m \geq 0$ immediately follows

$$\sum_{1 \leq k \leq n} \sum_{j \geq 0} A_{m,j} T^j(n,k) \ = \ \sum_{0 \leq k \leq n-1} \sum_{j \geq 0} A_{m,j} T^j(n,k)$$

(4) $A_{m,m},\ m = 0, 1, 2, ...$ are terms of A002457.

(5) For every $m \geq 0$

$$A_{m,0} \ = \ 1$$

(6) For each $m \geq 0$

$$\sum_{j \geq 0} A_{m,j} \ = \ \sum_{j \geq 0} \binom{2m+1}{j} \ - \ 1$$

$$\sum_{1 \leq k \leq n} \sum_{j \geq 0} A_{m,j} T^j(n,k) \ = \ n + \sum_{2 \leq k \leq n} \sum_{j \geq 1} A_{m,j} T^j(n,k)$$

(7) For each even power $2m,\ m \geq 0$ and $n \in \mathbb{Z}$ we have

$$\sum_{1 \leq k \leq n} \sum_{j \geq 0} \frac{1}{n} A_{m,j} T^j(n,k) \ = \ n^{2m}$$

(8) Forward and inverse summation identity

$$\sum_{1 \leq k \leq n} \sum_{j \geq 0} A_{m,j} T^j(n,k) = \sum_{1 \leq k \leq n} \sum_{j \geq 0} A_{m,m-j} T^{m-j}(n,k)$$

[3] expression_2_1.txt, [26].

3.2. **Example of use.** Recall existing pattern

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 6 & & & & & \\
 & & & & 1 & & 0 & & 30 & & & & \\
 & & & 1 & & -14 & & 0 & & 140 & & & \\
 & & 1 & & -120 & & 0 & & 0 & & 630 & & \\
 & 1 & & -1386 & & 660 & & 0 & & 0 & & 2772 & \\
\dots & & & & & & & & & & & &
\end{array} \tag{3.30}
$$

Figure 9. Triangle generated by $A_{m,j},\ 0 \leq j \leq m$.

By received formula $\sum_{k=0}^{n-1}\sum_{j\geq 0} A_{m,j}T^j(n,k) = n^{2m+1}$ each line of above triangle being multiplied by $T^j(n,k)$ and summed up to n or $n-1$ over k from 0 or 1, respectively, will result odd power of n, depending on which row of $A_{m,j},\ 0 \leq j \leq m$ is applied. Consider the case $n = 3,\ m = 2$, we introduce triangle built using $T(n,k),\ 1 \leq k \leq n$,

$$
\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & 1 & & 0 & & \\
 & 2 & & \underline{2} & & 0 & \\
3 & & 4 & & 3 & & 0
\end{array} \tag{3.31}
$$

Figure 10. Triangle generated by $T(n,k),\ 1 \leq k \leq n$, sequence A094053, [29] in OEIS.

Then,

$$
\begin{aligned}
3^{2\cdot 2+1} &= 1 + 0 \cdot 2^1 + 30 \cdot 2^2 \\
&+ 1 + 0 \cdot \underline{2}^1 + 30 \cdot \underline{2}^2 \\
&+ 1 + 0 \cdot 0^1 + 30 \cdot 0^2 \\
&= 121 + 121 + 1 = 243
\end{aligned}
$$

We've highlighted the terms of $A_{2,j}$ and $T(3,k)$ with different colors to be more easily to see regularity. Result we received are terms of the third row of triangle A300656.

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5. Conclusion

In this paper particular pattern, that is binomial distributed triangle A287326 in OEIS, which shows perfect cube n as sum of row terms over $0 \leq k \leq n-1$ or $1 \leq k \leq n$ is generalized. Firstly, we discussed analogs of A287326 for powers $2m+1 = 5, 7$, sequences A300656, A300785, respectively, then we derived coefficients $A_{m,j}$, such that for every $n \geq 0$ and $m \geq 0$ holds

$$\sum_{1 \leq k \leq n} \sum_{j \geq 0} A_{m,j} T^j(n,k) = n^{2m+1}$$

where $A_{m,j}$ is defined by definition (3.27). Therefore, question question (2.23) is answered positively. Section 3 is totally dedicated to complete and extended derivation of identity $\sum_{1 \leq k \leq n} \sum_{j \geq 0} A_{m,j} T^j(n,k) = n^{2m+1}$. Properties of triangle (2.14) and $L_m(n,k)$ are shown in properties 2.15 and subsection 3.1, respectively. Relation between Faulhaber's sum $\sum n^m$ and finite differences of power are shown in 2.3.

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