SERIES REPRESENTATION OF POWER FUNCTION

KOLOSOV PETRO

ABSTRACT. In this paper described numerical expansion of natural-valued power function x^n, in point $x = x_0$, where n, x_0 - positive integers. Applying numerical methods, the calculus of finite differences, particular pattern, that is sequence A287326 in OEIS, which shows the expansion of perfect cube n as row sum over $k,\ 0 \leq k \leq n-1$ is generalized, obtained results are applied to show expansion of monomial $n^{2m+1},\ m = 0, 1, 2, ..., \mathbb{N}$. Additionally, relation between Faulhaber's sum $\sum n^m$ and finite differences of power are shown in section 4.

2010 Math. Subject Class. 30BXX

ORCID: 0000-0002-6544-8880
e-mail: kolosovp94@gmail.com

Contents

1. Introduction and main results

In this paper particular pattern, that is triangle A287326 in OEIS, [11], which shows necessary items to expand perfect cube n as row sum is generalized and obtained results are applied on expansion of monomial $f(n) = n^m,\ (n, m) \in \mathbb{N}$. The coefficient $M_1(n, k)$ is k-th item of n-th row of triangle A287326, such that sum of the n-th row of A287326 over $0 \leq k \leq n-1$ is perfect cube. First, let review and basically describe Newton's Binomial Theorem, since our coefficient $M_1(n, k)$ is derived from finite difference of perfect cubes, which is taken regarding Binomial expansion. In elementary algebra, the Binomial theorem describes the algebraic expansion of powers of a binomial. The theorem describes expanding of the power

Date: May 9, 2018.
Key words and phrases. Power function, Binomial coefficient, Binomial Theorem, Finite difference, Perfect cube, Pascal's triangle, Series representation, Binomial Sum, Binomial identity, Binomial distribution, Hypercube.

of $(x+y)^n$ into a sum involving terms of the form ax^by^c where the exponents b and c are nonnegative integers with $b+c=n$, and the coefficient a of each term is a specific positive integer depending on n and b. The coefficient a in the term of ax^by^c is known as the Binomial coefficient. The main properties of the Binomial Theorem are next

Properties 1.1. *Binomial Theorem properties*

1. *The powers of x go down until it reaches $x_0 = 1$ starting value is n (the n in $(x+y)^n$)*
2. *The powers of y go up from 0 ($y^0 = 1$) until it reaches n (also n in $(x+y)^n$)*
3. *The n-th row of the Pascal's Triangle (see* [1], [12]*) will be the coefficients of the expanded binomial.*
4. *For each line, the number of products (i.e. the sum of the coefficients) is equal to $x+1$*
5. *For each line, the number of product groups is equal to 2^n*

According to the Binomial theorem, it is possible to expand any power of $x+y$ into a sum involving Binomial coefficients

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \tag{1.2}$$

Let expand monomial $f(x) = x^n$, where x and n are positive integers, applying a finite difference operator

Lemma 1.3. *Power function could be represented as discrete integral of its first order finite difference*

$$\begin{aligned} x^n &= \sum_{k=0}^{x-1} \underbrace{nk^{n-1}h + \binom{n}{2}k^{n-2}h^2 + \cdots + \binom{n}{n-1}kh^{n-1} + h^n}_{\Delta_h[x^n]=(x+h)^n - x^n} \\ &= \sum_{j=0}^{x-1} \sum_{k=1}^{n} \binom{n}{k} j^{n-k} h^k, \ \ x, n \in \mathbb{N}, \ \ h \in \mathbb{R} \end{aligned}$$

Or, by means of Fundamental Theorem of Calculus

$$x^n = \int\limits_0^x nt^{n-1}dt = \sum_{k=0}^{x-1} \int_k^{k+1} nt^{n-1}dt = \sum_{k=0}^{x-1} (k+1)^n - k^n$$

Let describe the derivation of the sequence A287326 in OEIS, which shows the expansion of perfect cube n as row sum over k, $0 \leq k \leq n-1$. First, review a difference table of perfect cubes ([4], eq. 7)

(1.4)

n	$\Delta^0(n^3)$	$\Delta^1(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	6
6	216	127	42	6
7	343	169	48	6
8	512	217	54	
9	729	271		
10	1000			

Table 1: Difference table of perfect cubes $n,\ 0 \leq n \leq 10$ of order $k,\ 0 \leq k \leq 3$.

Note that increment h is set to be $h = 1$ and $k > 2$-order difference is taken regarding to [8], [6]. Reviewing Figure (1.4), we have noticed that

$$
\begin{aligned}
\Delta(0^3) &= 1 + 3! \cdot 0 \\
\Delta(1^3) &= 1 + 3! \cdot 0 + 3! \cdot 1 \\
\Delta(2^3) &= 1 + 3! \cdot 0 + 3! \cdot 1 + 3! \cdot 2 \\
\Delta(3^3) &= 1 + 3! \cdot 0 + 3! \cdot 1 + 3! \cdot 2 + 3! \cdot 3 \\
&\vdots \\
\Delta(n^3) &= 1 + 3! \cdot 0 + 3! \cdot 1 + 3! \cdot 2 + 3! \cdot 3 + \cdots + 3! \cdot n
\end{aligned}
\tag{1.5}
$$

Also, another interesting thing was observed, according to Donald Knuth, the perfect cube n is

$$
n^3 = 6\binom{n+1}{3} + \binom{n}{1}
\tag{1.6}
$$

Returning to expression (1.5), we can find that

$$
\Delta(n^3) = 6\binom{n+1}{2} + \binom{n}{0}
\tag{1.7}
$$

Then

$$
\{\Delta^2(n^3)\}_{n=0}^t = \left\{ 6\binom{n+1}{1} + \underbrace{\binom{n}{-1}}_{=0} \right\}_{n=0}^t = \{6, 12, 18, 24, ..., 6t\}
\tag{1.8}
$$

which fit according to $\Delta^2(n^3)$ values from Figure (1.4). Therefore, we represent perfect cube n as

$$
n^3 = (1 + 3! \cdot 0) + (1 + 3! \cdot 0 + 3! \cdot 1) + \cdots + (1 + 3! \cdot 0 + \cdots + 3! \cdot (n-1))
\tag{1.9}
$$

Generalizing above expression, we have

$$
n^3 = n + (n-0) \cdot 3! \cdot 0 + (n-1) \cdot 3! \cdot 1 + \cdots + (n-(n-1)) \cdot 3! \cdot (n-1)
\tag{1.10}
$$

Provided that n is natural. Now we apply a compact sigma notation on (1.10), hereby

$$n^3 = n + \sum_{k=0}^{n-1} 3! \cdot nk - 3! \cdot k^2 \tag{1.11}$$

As sum $\sum_{k=0}^{n-1} 3! \cdot nk - 3! \cdot k^2$ consists of n terms, we have right to move n from (1.11) under sigma notation, we get

$$n^3 = \sum_{k=0}^{n-1} 3! \cdot nk - 3! \cdot k^2 + 1 \tag{1.12}$$

Property 1.13. *Let be a sets $A(n) := \{1,\ 2,\ \dots,\ n\}$, $B(n) := \{0,\ 1,\ \dots,\ n\}$, $C(n) := \{0,\ 1,\ \dots,\ n-1\}$, let be expression (1.12) defined as*

$$T(n,\ C(n)) := \sum_{k \in C(n)} 3! \cdot nk - 3! \cdot k^2 + 1$$

where x is natural-valued variable and $C(n)$ is iteration set of (1.12), then we have equality

$$T(n,\ A(n)) = T(n,\ C(n)) \tag{1.14}$$

Let review and define expression (1.10) as

$$U(n,\ C(n)) := n + 3! \cdot \sum_{k \in C(n)} nk - k^2$$

then

$$U(n,\ A(n)) = U(n,\ B(n)) = U(n,\ C(n)) \tag{1.15}$$

Other words, changing of iteration sets of(1.10) and (1.12) by $A(n),\ B(n),\ C(n)$ and $A(n),\ C(n)$, respectively, doesn't change resulting value for each natural x.

Proof. Let be a plot $y(n,k) = 3! \cdot nk - 3! \cdot k^2 + 1,\ k \in \mathbb{R},\ 0 \le k \le 10$, given $n = 10$

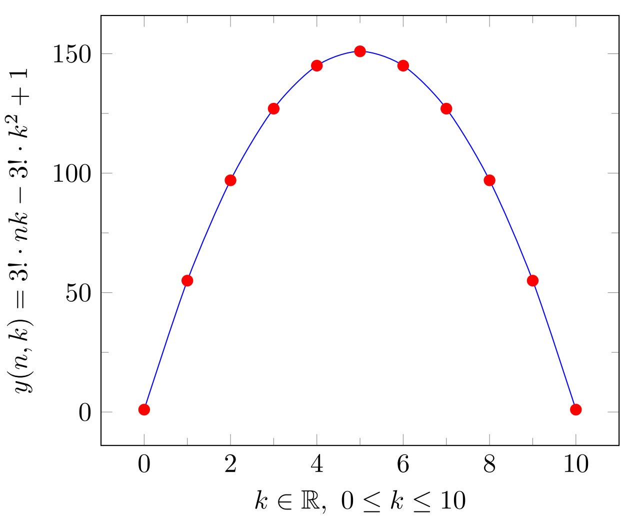

Figure 2. Plot of $y(n,k) = 3! \cdot nk - 3! \cdot k^2 + 1,\ k \in \mathbb{R},\ 0 \le k \le n$, where $n = 10$.

Obviously, being a parabolic function, it's symmetrical over $\frac{n}{2}$, hence equivalent $T(n,\ A(n)) = T(n,\ C(n))$ follows. Reviewing (1.10) and denote $u(n,t) = n^2t - t^2$, we can conclude, that $u(n,0) = u(n,n) = 0$, then equality of $U(n,\ A(n)) = U(n,\ B(n)) = U(n,\ C(n))$ immediately follows. This completes the proof. □

Review above property (1.13). Let be an example of triangle built using

$$y(n,k) = 3! \cdot nk - 3! \cdot k^2 + 1,\ 0 \le k \le n \tag{1.16}$$

over n from 0 to $n = 4$, where n denotes corresponding row and k shows the item of row n.

(1.17)

Row 0:					1				
Row 1:				1		1			
Row 2:			1		7		1		
Row 3:		1		13		13		1	
Row 4:	1		19		25		19		1

Figure 3. Triangle generated by (1.16) from 0 to $n = 4$, sequence A287326 in OEIS, [11].

Note that n-th row sum of Triangle (1.17) over $0 \le k \le n-1$ returns perfect cube n. We can see that each row with respect to variable $n = 0,\ 1,\ 2,\ 3,\ 4, \ldots$, has Binomial distribution of row terms. One could compare Triangle (1.17) with Pascal's triangle [1], [12]

Row 0:					1				
Row 1:				1		1			
Row 2:			1		2		1		
Row 3:		1		3		3		1	
Row 4:	1		4		6		4		1

Figure 4. Pascal's triangle up to forth row, sequence A007318 in OEIS.

1.1. **Properties of Triangle (1.17) and other expansions.** Review the triangle (1.17), define the k-th, $0 \le k \le n$, item of n-th row of triangle as

Definition 1.18.

$$M_1(n,k) := 3! \cdot nk - 3! \cdot k^2 + 1 = 6(n-k)k + 1,\ 0 \le k \le n \tag{1.19}$$

Let us approach to show a few properties of triangle (1.17) and $M_1(n,k)$.

Properties 1.20. *Properties of triangle (1.17).*

(1) *Summation of items $M_1(n,k)$ of n-th row of triangle (1.17) over k from* 0 *to $n-1$ returns perfect cube n as binomial of the form*

$$\sum_{k=0}^{n-1} M_1(n,k) = A_{0,n}n - B_{0,n} = n^3,\ n \ge 0 \tag{1.21}$$

Since the property (1.14) holds, (1.21) could be rewritten as

$$\sum_{k=1}^{n} M_1(n,k) = A_{1,n}n - B_{1,n} = n^3,\ n \geq 0 \tag{1.22}$$

where $A_{\overline{0,1},n}$ *and* $B_{\overline{0,1},n}$ *- integers* ***depending*** *on variable* $n \in \mathbb{N}$ *and on sets* $U(n),\ S(n)$*, respectively. Note that* $A_{0,n}n \neq A_{1,n}n,\ B_{0,n} \neq B_{1,n}$

(2) *Relation between* $A_{0,n}$ *and* $A_{1,n}$

$$A_{0,n+1} = A_{1,n},\ n \geq 1$$

.

(3) *Summation of items* $M_1(n,k)$ *of* n*-th row of triangle (1.17) over* k *from* 0 *to* n *returns* n^3+1

$$\sum_{k=0}^{n} M_1(n,k) = n^3 + 1 \tag{1.23}$$

(4) *First item of each row's number corresponding to central polygonal numbers sequence* $a(n) = \frac{n^2+n+2}{2}$ *(sequence A000124 in OEIS,* [13]*) returns finite difference of consequent perfect cubes. For example, let be a* k*-th row of triangle (1.17), such that* $k = \frac{n^2+n+2}{2},\ n = 0,1,2,...,$ *then item*

$$M_1\left(\frac{n^2+n+2}{2}, 1\right) = (n+1)^3 - n^3,\ n \geq 0 \tag{1.24}$$

(5) *Items of (1.17) have Binomial distribution over rows.*

(6) *The linear recurrence, for any* k *and* $n > 0$

$$2M_1(n,k) = M_1(n+1,k) + M_1(n-1,k) \tag{1.25}$$

This linear recurrence is direct result of second order binomial transform of $M_1(n,k)$ *over* n*.*

(7) *Linear recurrence, for each* $n > k$

$$2M_1(n,k) = M_1(2n-k,k) + M_1(2n-k,0) \tag{1.26}$$

(8) *From (1.24) follows that*

$$\sum_{k=0}^{n-1} M_1(n,k) = \sum_{k=0}^{n-1} M_1\left(\frac{n^2+n+2}{2}, 1\right) = n^3,\ n \geq 0 \tag{1.27}$$

(9) *Triangle (1.17) is symmetric, i.e*

$$M_1(n,k) = M_1(n,n-k) \tag{1.28}$$

As its noticed in (1.21), summation of each n-th row of Triangle (1.17) from 0 to $n-1$ returns perfect cube n, then, by properties (1.24), (1.25), (1.26), for each positive integer n the $n^m,\ m = 0,1,2,...$ could be found via multiplication of each term of (1.21) by n^{m-3}

$$
\begin{aligned}
(1.29) \quad n^m &= \sum_{k=0}^{n-1} M_1(n,k) n^{m-3} = \frac{1}{2} \sum_{k=0}^{n-1} \left[M_1(n+1,k) + M_1(n-1,k)\right] n^{m-3} \\
&= \sum_{k=0}^{n-1} \frac{1}{2} \left[M_1(2n-k,k) + M_1(2n-k,0)\right] n^{m-3} \\
&= \sum_{k=0}^{n-1} \frac{1}{2} M_1 \left(\frac{n^2+n+2}{2}, 1 \right) n^{m-3} \\
&= \sum_{k=0}^{n-1} \frac{1}{2} \left[M_1 \left(\frac{n^2+n}{2}, 1 \right) + M_1 \left(\frac{n^2+n+4}{2}, 1 \right) \right] n^{m-3} \\
&= \sum_{k=0}^{n-1} \frac{1}{2} \left[M_1 \left(\binom{m+1}{2}, 1 \right) + M_1 \left(\binom{m+1}{2} + \binom{2}{1}, 1 \right) \right] n^{m-3}
\end{aligned}
$$

To show other representation of monomial $n^m, \ m = 0, 1, 2, ...$, review (1.11), let move n in $n + 3! \sum_k nk - k^2$ under the sum operator and change iteration set from $\{0, \ n-1\}$ to $\{1, \ n-1\}$, then we get

$$
n^3 = \sum_{k=1}^{n-1} 3! \cdot nk - 3! \cdot k^2 + \frac{n}{(n-1)}, \ n \neq 1 \tag{1.30}
$$

Review right part of (1.30), let the term $\frac{n}{n-1}$ be written as $\frac{n}{n-1} = \frac{n+1-1}{n-1} = 1 + \frac{1}{n-1}$, given the power $m > 3$, multiplying each term of (1.30) by n^{m-3} we can observe that

$$
n^m - 1 = \sum_{k=1}^{n-1} M_1(n,k) n^{m-3} + n^{m-4} + n^{m-5} + \cdots + n + 1 \tag{1.31}
$$

Applying properties (1.24), (1.25), (1.26), we can rewrite (1.31) as

$$
\begin{aligned}
n^m - 1 &= \sum_{k=1}^{n-1} \frac{1}{2} \left[M_1(2n-k,k) + M_1(2n-k,0)\right] n^{m-3} + n^{m-4} + \cdots + n + 1 \\
(1.32) \quad &= \sum_{k=1}^{n-1} \frac{1}{2} \left[M_1(n+1,k) + M_1(n-1,k)\right] n^{m-3} + n^{m-4} + \cdots + n + 1 \\
&= \sum_{m=0}^{n-1} \frac{1}{2} \left[M_1 \left(\tfrac{n^2+n}{2}, 1 \right) + M_1 \left(\tfrac{n^2+n+4}{2}, 1 \right) \right] n^{m-3} + n^{m-4} + \cdots + n + 1 \\
&= \sum_{k=1}^{n-1} M_1 \left(\tfrac{n^2+n+2}{2}, 1 \right) n^{m-3} + n^{m-4} + \cdots + 1
\end{aligned}
$$

Review (1.31), let move 1 from left part of (1.31) under sum operator in the right part, therefore we add a term $\frac{1}{n-1}$ to initial function $M_1(n,k)n^{m-3} + n^{m-4} + n^{m-5} + \cdots + n + 1$. By means of expansion $\frac{1}{1-n} = -\frac{1}{n-1} = 1 + n + n^2 + n^3 + \cdots$, the (1.31) could be rewritten respectively

$$
n^m = \sum_{m=1}^{n-1} M_1(n,m) n^{m-3} + n^{m-4} + \cdots + n + 1 - 1 - n^2 - n^3 - \cdots \tag{1.33}
$$

Particularizing (1.33) we have

$$\begin{aligned}(1.34)\quad n^m &= \sum_{m=1}^{n-1} \left[M_1(n,m)-1\right] n^{m-3} - n^{m-2} - n^{m-1} - \cdots \\ &= \sum_{m=1}^{n-1} \left[M_1\left(\frac{n^2+n+2}{2},1\right)-1\right] n^{m-3} - n^{m-2} - n^{m-1} - \cdots\end{aligned}$$

Hereby, the follow question is stated

Question 1.35. *Has the Triangle (1.17) any analogs in order to receive monomial* $x^t\ t>3$ *as row sum? Is it exist* $M_v(n,k),\ v\neq 1$*, such that*

$$\sum_{k=0}^{n-1} M_v(n,k) = n^t,\ v \neq t\ ?$$

2. Generalization of sequence A287326

Considering the OEIS sequences A300656 and A300785, (question 1.31) can be answered positively, since the sequences A300656 and A300785 are analogs of Triangle (1.17), that show fifth and seventh powers as row sums. To generalize our sequence A287326 for each odd power $2m+1,\ m=0,1,2...$ we have to review the generating formulas of sequences A287326, A300656, A300785 as

$$\sum_{k=0}^{n-1}\sum_{j=0}^{m} A_{m,j}(n-k)^j k^j = n^{2m+1},\ m=1,2,3 \tag{2.1}$$

Where $A_{m,j}$ is unknown. For example, generating functions of our sequences A287326, A300656, A300785 are

$$\begin{cases} 6k(n-k)+1, & \text{for } A287326 \\ 30k^2(n-k)^2+1, & \text{for } A300656 \\ 140k^3(n-k)^3-14k(n-k)+1, & \text{for } A300785 \end{cases} \tag{2.2}$$

Reviewing (2.2) we observe that coefficients are $\{6,1\},\ \{30,0,1\},\ \{140,0,-14,1\}$ in each generating function of A287326, A300656, A300785, respectively. To generalize above results over odd powers $2m+1,\ m=0,1,2...$ one has to solve the system of equations. Define the part of (2.1) as

$$M_m(n,k) := \sum_{j=0}^{m} A_{m,j}(n-k)^j k^j,\ m\geq 0 \tag{2.3}$$

Hereby let be a system

$$\begin{cases} M_m(1,0) = 1^{2m+1} \\ M_m(2,0)+M_m(2,1) = 2^{2m+1} \\ M_m(3,0)+M_m(3,1)+M_m(3,2) = 3^{2m+1} \\ \qquad\vdots \\ M_m(t,0)+M_m(t,1)+\cdots+M_m(t,t-1) = t^{2m+1},\ t\geq m \end{cases} \tag{2.4}$$

Solving system (2.4) we receive a set of coefficients $\{A_{m,0},\ \dots,\ A_{m,m}\}$ for each particular t, such that (2.1) holds. List of solutions[1] of system (2.4) is split and assigned to OEIS under the numbers A302971 (numerators of $A_{m,j}$) and A304042 (denominators of $A_{m,j}$). To reach recurrent formula of $A_{m,j}$, first let fix the unused values $A_{m,j} = 0$, for $j < 0$ or $j > m$, so we don't need to care about the summation range for j, then by expanding $(n-k)^j$ and using Faulhaber's formula, we get

$$\begin{aligned}(2.5)\quad \sum_{k=0}^{n-1}(n-k)^j k^j &= \sum_{k=0}^{n-1}\sum_{i}^{\infty}\binom{j}{i} n^{j-i}(-1)^i k^{i+j} \\ &= \sum_{i}^{\infty}\binom{j}{i} n^{j-i}\frac{(-1)^i}{i+j+1}\left[\sum_{t}^{\infty}\binom{i+j+1}{t} B_t n^{i+j+1-t} - B_{i+j+1}\right] \\ &= \sum_{i,t}^{\infty}\binom{j}{i}\frac{(-1)^i}{i+j+1}\binom{i+j+1}{t} B_t n^{2j+1-t} - \sum_{i}^{\infty}\binom{j}{i}\frac{(-1)^i}{i+j+1} B_{i+j+1} n^{j-i}\end{aligned}$$

where B_t are Bernoulli numbers [14]. Now, we notice that

$$(2.6)\qquad \sum_{i}^{\infty}\binom{j}{i}\frac{(-1)^i}{i+j+1}\binom{i+j+1}{t} = \begin{cases}\frac{1}{(2j+1)\binom{2j}{j}}, & \text{if } t=0;\\ \frac{(-1)^j}{t}\binom{j}{2j-t+1}, & \text{if } t>0\end{cases}$$

In particular, the last sum is zero for $0 < t \le j$. To combine or cancel identical terms across the two sums from right part of (2.5) more easily we introducing $\ell = 2j+1-t$ and $\ell = j-i$, respectively, we get

$$\begin{aligned}(2.7)\qquad \sum_{k=0}^{n-1}(n-k)^j k^j &= \frac{1}{(2j+1)\binom{2j}{j}} n^{2j+1} + \sum_{\ell=-\infty}^{\infty}\frac{(-1)^j}{2j+1-\ell}\binom{j}{\ell} B_{2j+1-\ell} n^{\ell} \\ &\quad - \sum_{\ell=-\infty}^{\infty}\binom{j}{\ell}\frac{(-1)^{j-\ell}}{2j+1-\ell} B_{2j+1-\ell} n^{\ell} \\ &= \frac{1}{(2j+1)\binom{2j}{j}} n^{2j+1} + 2\sum_{\text{odd } \ell}^{\infty}\frac{(-1)^j}{2j+1-\ell}\binom{j}{\ell} B_{2j+1-\ell} n^{\ell}.\end{aligned}$$

Note that binomial coefficient is defined as

$$\binom{n}{k} = \begin{cases}\frac{n}{k!(n-k)!}, & \text{if } 0 \le k \le n;\\ 0, & \text{otherwise}\end{cases}$$

Therefore, we don't use strict limits on sums of above derivation as it's much easier to review each sum as summing from $-\infty$ to $+\infty$ (unless specified otherwise), where only a finite number of terms are nonzero. Now, using the definition of $A_{m,j}$ we obtain the following identity for polynomials in n

$$\begin{aligned}(2.8)\qquad &\sum_{j}^{\infty} A_{m,j}\frac{1}{(2j+1)\binom{2j}{j}} n^{2j+1} + 2\sum_{j,\ \text{odd } \ell}^{\infty} A_{m,j}\binom{j}{\ell}\frac{(-1)^j}{2j+1-\ell} B_{2j+1-\ell} n^{\ell} \\ &\equiv n^{2m+1}\end{aligned}$$

[1]One can produce a list of solutions of system (2.4) up to $t = 11$ using Mathematica code solutions_system_2_4.txt, [24].

Taking the coefficient of n^{2m+1} in above expression, we get $A_{m,m} = (2m+1)\binom{2m}{m}$, and taking the coefficient of x^{2d+1} for an integer d in the range $m/2 \leq d < m$ we get $A_{m,d} = 0$. Taking the coefficient of n^{2d+1} in (2.8) for $m/4 \leq d < m/2$, we get

$$A_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} + 2(2m+1)\binom{2m}{m}\binom{m}{2d+1}\frac{(-1)^m}{2m-2d}B_{2m-2d} = 0, \tag{2.9}$$

i.e

$$A_{m,d} = (-1)^{m-1}\frac{(2m+1)!}{d!d!m!(m-2d-1)!}\frac{1}{m-d}B_{2m-2d}. \tag{2.10}$$

Continue similarly, we can express $A_{m,j}$ for each integer j in range $m/2^{s+1} \leq j < m/2^s$ (iterating consecutively $s = 1, 2, ...$) via previously determined values of $A_{m,d},\ d < j$ as follows

$$A_{m,j} = (2j+1)\binom{2j}{j}\sum_{d=2j+1}^{m} A_{m,d}\binom{d}{2j+1}\frac{(-1)^{d-1}}{d-j}B_{2d-2j}. \tag{2.11}$$

The same formula holds also for $d = 0$. Hereby, we have shown that (2.1) holds for each $m \geq 0$.

Definition 2.12. We define here a generalized sequence of coefficients $A_{m,j}$, such that $\sum_{k=0}^{n-1}\sum_{j=0}^{m} A_{m,j}(n-k)^j k^j = n^{2m+1},\ n \geq 0,\ m = 0, 1, 2, ...$

$$A_{m,j} := \begin{cases} 0, & \text{if } j < 0 \text{ or } j > m \\ (2j+1)\binom{2j}{j}\sum_{d=2j+1}^{m} A_{m,d}\binom{d}{2j+1}\frac{(-1)^{d-1}}{d-j}B_{2d-2j}, & \text{if } 0 \leq j < m \\ (2j+1)\binom{2j}{j}, & \text{if } j = m \end{cases}$$

First five rows of triangle generated by $A_{m,j}$ are

$$\begin{array}{cccccccccccc} &&&&& 1 &&&&& \\ &&&& 1 && 6 &&&& \\ &&& 1 && 0 && 30 &&& \\ && 1 && -14 && 0 && 140 && \\ & 1 && -120 && 0 && 0 && 630 & \\ 1 && -1386 && 660 && 0 && 0 && 2772 \\ \dots &&&&&&&&&& \end{array} \tag{2.13}$$

Figure 5. Triangle generated by $A_{m,j},\ 0 \leq j \leq m$.

Note that starting from row $m \geq 11$ the terms of Triangle (2.13) consist rational numbers, for example, $A_{11,1} = 800361655623.60$, therefore it's split into two sequences A302971, A304042 that show numerators and denominators of $A_{m,j},\ 0 \leq j \leq m$, respectively. To verify the terms that definition (2.12) produces one should refer to Mathematica code[2].

[2] def_2_12.txt, [25]

2.1. **Properties of $M_m(n,k)$ and $A_{m,j}$.** Here we show a few properties of definition $M_m(n,k)$, some of them correlates with properties of $M_1(n,k)$ in subsection (1.1)

(1) Sum of $A_{m,j},\ m \geq 0$ gives

$$\sum_j A_{m,j} \ = \ 2^{2m+1} - 1 \tag{2.14}$$

(2) Generalization of (2.1) for all positive integers $m \geq 0$

$$\sum_{k=0}^{n-1} \sum_j A_{m,j} (n-k)^j k^j \ = \ n^{2m+1}$$

Can be verified via Mathematica code[3].

(3) Similarly to particular property (1.28), items of $\{M_m(n,k)\}_{k=0}^{n},\ m \geq 0$ is symmetric, i.e

$$M_m(n,k) \ = \ M_m(n,n-k) \tag{2.15}$$

(4) From (2.15) immediately follows

$$\sum_{k=0}^{n-1} M_m(n,k) \ = \ \sum_{k=1}^{n} M_m(n,k) = n^{2m+1},\ n \geq 0,\ m \geq 0 \tag{2.16}$$

(5) Generalization of linear recurrence (1.25), that is $2M_1(n,k) = M_1(n+1,k) + M_1(n-1,k)$

$$\sum_{k=0}^{t} (-1)^k \binom{t+1}{k} M_m(n+t-k,k) = 0,\ n \geq 0,\ t \geq m \tag{2.17}$$

then

$$\binom{t+1}{t} M_m(n,k) = \begin{cases} \sum_{k=0}^{t-1} (-1)^{k+1} \binom{t+1}{k} M_m(n+t-k,k), & t = \text{even} \\ \sum_{k=0}^{t-1} (-1)^{k} \binom{t+1}{k} M_m(n+t-k,k), & t = \text{odd} \end{cases} \tag{2.18}$$

(6) $A_{m,m},\ m = 0,1,2,...$ are terms of A002457.

2.2. **Generalized Binomial Series by means of properties (1.21), (1.22).** Reviewing properties (1.21) and (1.22), we can say that for each natural n holds

$$n^m = A_{\overline{0,1},n} n^{m-2} - B_{\overline{0,1},n} n^{m-3} \tag{2.19}$$

Note that $A_{\overline{0,1},n}$ and $A_{m,j}$ are different definitions. Rewrite the right part of (2.19) regarding to itself as recursion

$$\begin{aligned} n^m &= A_{\overline{0,1},n}(A_{\overline{0,1},n} n^{m-4} - B_{\overline{0,1},n} n^{m-5}) - B_{\overline{0,1},n}(A_{\overline{0,1},n} n^{m-5} - B_{\overline{0,1},n} n^{m-6}) \\ &= A_{\overline{0,1},n}^2 n^{m-4} - 2A_{\overline{0,1},n} B_{\overline{0,1},n} n^{m-5} + B_{\overline{0,1},n}^2 n^{m-6} \end{aligned}$$

Reviewing above expression we can observe Binomial coefficients before each $A_{\overline{0,1},n} \cdot B_{\overline{0,1},n}$. Continuous j-times recursion of right part of (2.19) gives us

$$n^m = \sum_{k=0}^{\infty} (-1)^k \binom{j}{k} A_{\overline{0,1},n}^{j-k} B_{\overline{0,1},n}^{k} n^{m-2j-k},\ j \geq 0 \tag{2.20}$$

Solutions $A_{\overline{0,1},n},\ B_{\overline{0,1},n}$ of equation (2.19) are listed in follow table

[3] expression_2_1.txt, [26].

x	$A_{0,x}$	$B_{0,x}$	$A_{1,x}$	$B_{1,x}$
1	1	0	6	5
2	6	4	18	28
3	18	25	36	81
4	36	80	60	176
5	60	175	90	325
6	90	324	126	540
7	126	539	168	833
8	168	832	216	1216
9	216	1215	270	1701
10	270	1700	330	2300

Table 9. Array of coefficients $A_{\overline{0,1},n}$, $B_{\overline{0,1},n}$ given $n = 1, ..., 10$.

Sequence $A_{1,x}$ is generated by $3n^2 + 3n$, sequence A028896 in OEIS, [23]. Sequence $B_{1,n}$ is generated by $2n^3 + 3n^2$, sequence A275709 in OEIS, [20].

3. Relation between Pascal's Triangle and Hypercubes

In this section let review and generalize well known fact about connection between row sums of Pascal triangle and $2-$dimension Hypercube, recall property

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \tag{3.1}$$

Now, let multiply each k-th term of of n-th row of Pascal's triangle [1] by 2^k

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 2 & & & \\ & & 1 & & 4 & & 4 & & \\ & 1 & & 6 & & 12 & & 8 & \\ 1 & & 8 & & 24 & & 32 & & 16 \end{array}$$

Figure 10. Triangle built by $\binom{n}{k} \cdot 2^k$, $0 \le k \le n \le 4$.

We can notice that

$$\sum_{k=0}^{n} \binom{n}{k} \cdot 2^k = 3^n, \quad 0 \le k \le n, \quad (n,\ k) \in \mathbb{N} \tag{3.2}$$

Hereby, let be theorem

Theorem 3.3. *Volume of n-dimension hypercube with length m could be calculated as*

$$m^n = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} (-1)^{k-j} m^j \tag{3.4}$$

where m and n - positive integers.

Proof. Recall induction over m, in (3.1) is shown a well-known example for $m = 2$.

$$2^n = \sum_{k=0}^{n} \binom{n}{k} (2-1)^k \tag{3.5}$$

Review (3.5) and suppose that

$$(\underbrace{2+1}_{m=3})^n = \sum_{k=0}^{n} \binom{n}{k} (\underbrace{(2-1)+1}_{m-1})^k \tag{3.6}$$

And, obviously, this statement holds by means of Newton's Binomial Theorem [2], [3] given $m = 3$, more detailed, recall expansion for $(x+1)^n$ to show it.

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \tag{3.7}$$

Substituting $x = 2$ to (3.7) we have reached (3.6).
Next, let show example for each $m \in \mathbb{N}$. Recall Binomial theorem to show this

$$m^n = \sum_{k=0}^{n} \binom{n}{k} (m-1)^k \tag{3.8}$$

Hereby, for $m+1$ we receive Binomial theorem again

$$(m+1)^n = \sum_{k=0}^{n} \binom{n}{k} m^k \tag{3.9}$$

Review result from (3.8) and substituting Binomial expansion $\sum_{j=0}^{k} \binom{k}{j} (-1)^{n-k} m^j$ instead $(m-1)^k$ we receive desired result

$$\begin{aligned} m^n &= \sum_{k=0}^{n} \binom{n}{k} \underbrace{(m-1)^k}_{\sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} m^j} = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} m^j \\ &= \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} (-1)^{k-j} m^j \end{aligned} \tag{3.10}$$

This completes the proof. □

The (3.5) is analog of MacMillan Double Binomial Sum (see equation 13 in [5]).

4. Faulhaber's formula and finite differences

In this section we try to go into details about our identities (1.6), (1.7), (1.8). On the page 9 of [21] we can find an identities

$$\begin{aligned} n &= \binom{n}{1} \\ n^3 &= 6\binom{n+1}{3} + \binom{n}{1} \\ n^5 &= 120\binom{n+2}{5} + 30\binom{n+1}{3} + \binom{n}{1} \end{aligned}$$

For example, consider a first order finite difference applying above identities, we have

$$\begin{aligned} \Delta n &= \binom{n}{0} \\ \Delta n^3 &= 6\binom{n+1}{2} + \binom{n}{0} \\ \Delta n^5 &= 120\binom{n+2}{4} + 30\binom{n+1}{2} + \binom{n}{0} \end{aligned}$$

These identities may have the view

$$n^m = \begin{cases} \sum_k J(n,k)\binom{n+m-k}{2m+1-2k}, & m = \text{odd} \\ \sum_k J(n,k)n\binom{n+m-k}{2m+1-2k}, & m = \text{even} \end{cases} \tag{4.1}$$

where $J(n,k)$ is defined by same identity. Particularly, coefficients $J(n,k)$ are related to what Riordan ([22], page 213) has called *central factorial numbers of the second kind*. The t-order finite difference of monomial n^m is

$$\Delta^t n^m = \begin{cases} \sum_k J(n,k)\binom{n+m-k}{2m+1-t-2k}, & m = \text{odd} \\ \sum_k J(n,k)n\binom{n+m-k}{2m+1-t-2k}, & m = \text{even} \end{cases} \tag{4.2}$$

5. Acknowledgements

We would like to thank to Dr. Max Alekseyev (Department of Mathematics and Computational Biology, George Washington University), Dr. Hansruedi Widmer, Dr. Ron Knott (Visiting Fellow, Dept of Mathematics at University of Surrey) and Mr. Albert Tkaczyk for very useful comments, suggestions and help during writing of this article.

6. Conclusion

In this paper particular pattern, that is triangle A287326 in OEIS, which shows the expansion of perfect cube n as row sum over $0 \leq k \leq n-1$ is generalized for each odd power $2m+1,\ m = 0,1,2...$, we have reached a result

$$\sum_{k=0}^{n-1}\sum_{j=0}^{m} A_{m,j}(n-k)^j k^j = n^{2m+1},\ m = 0,1,2,...,$$

where $A_{m,j}$ is defined by definition (2.12). The coefficient $M_1(n,k)$ is defined by definition (1.18) and generalized to $M_m(n,k),\ m > 1$ at section 2. Properties of $M_1(n,k)$ and $M_m(n,k)$ are shown in (1.20) and subsection 2.1, respectively. An analog of MacMillan Double Binomial Sum [5] is shown in section 3. Relation between Faulhaber's sum $\sum n^m$ and finite differences of power are shown in section 4.

References

[1] Conway, J. H. and Guy, R. K. "Pascal's Triangle." In The Book of Numbers. New York: Springer-Verlag, pp. 68-70, 1996.
[2] Abramowitz, M. and Stegun, I. A. (Eds.). Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. New York: Dover, pp. 10, 1972.
[3] Arfken, G. Mathematical Methods for Physicists, 3rd ed. Orlando, FL: Academic Press, pp. 307-308, 1985.
[4] Weisstein, Eric W. "Finite Difference." From Mathworld.
[5] Weisstein, Eric W. "Power." From Mathworld.
[6] Richardson, C. H. An Introduction to the Calculus of Finite Differences. p. 5, 1954.
[7] Kevin J. McGown, Harold R. Parks. The generalization of Faulhabers formula to sums of non-integral powers, available online at https://doi.org/10.1016/j.jmaa.2006.08.019, 2006.
[8] Bakhvalov N. S. Numerical Methods: Analysis, Algebra, Ordinary Differential Equations p. 59, 1977. (In russian)
[9] The OEIS Foundation Inc.,The On-Line Encyclopedia of Integer Sequences, 1964-present https://oeis.org/
[10] N. J. A. Sloane et al., Entry "Coordination sequence for hexagonal lattice", A008458 in [9], 2002-present.

[11] Petro, Kolosov, et al., Entry "Triangle read by rows: $T(n,k) = 6*(n-k)*k+1$", A287326 in [9], 2017.
[12] N. J. A. Sloane and Mira Bernstein et al., "Pascal's Triangle", Entry A007318 in [9], 1994-present.
[13] N. J. A. Sloane. "Central polygonal numbers" Entry A000124 in [9], 1994-present.
[14] Weisstein, Eric W. "Bernoulli Number." From Mathworld–A Wolfram Web Resource.
[15] Petro, Kolosov, et al., Entry "Triangle read by rows: $T(n,k) = 30*(n-k)^2*k^2+1$", A300656 in [9], 2018.
[16] Petro, Kolosov, et al., Entry "Triangle read by rows: $T(n,k) = 140*(n-k)^3*k^3*-14*(n-k)*k+1$", A300785 in [9], 2018.
[17] Petro, Kolosov, et al., Entry "Triangle read by rows: Numeator$(A_{m,j}),\ 0 \leq j \leq m,\ m \geq 0$", A302971 in [9], 2018.
[18] Petro, Kolosov, et al., Entry "Triangle read by rows: Denominator$(A_{m,j}),\ 0 \leq j \leq m,\ m \geq 0$", A304042 in [9], 2018.
[19] N. J. A. Sloane, et al., Entry "$a(n) = (2n+1)!/n!^2$", A002457 in [9].
[20] Joshua Giambalvo, et al., Entry "$a(n) = 2*n^3+3*n^2$", A275709 in [9], 2016.
[21] Donald E. Knuth., Johann Faulhaber and Sums of Powers, pp. 9-10., 1992. arXiv preprint, https://arxiv.org/abs/math/9207222v1.
[22] John Riordan, Combinatorial Identities (New York: John Wiley & Sons, 1968).
[23] Joe Keane, et al., Entry "6 times triangular numbers: $a(n) = 3*n*(n+1)$", A028896 in [9], 1999.
[24] https://kolosovpetro.github.io/mathematica_codes/solutions_system_2_4.txt
[25] https://kolosovpetro.github.io/mathematica_codes/def_2_12.txt
[26] https://kolosovpetro.github.io/mathematica_codes/expression_2_1.txt