SERIES REPRESENTATION OF POWER FUNCTION

KOLOSOV PETRO

Abstract. In this paper described numerical expansion of natural-valued power function x^n, in point $x = x_0$, where n, x_0 - positive integers. Applying numerical methods, that is calculus of finite differences, particular pattern, that is sequence A287326 in OEIS, which shows us necessary items to expand $x^3,\ x \in \mathbb{N}$ is reached and generalized, obtained results are applied to show expansion of power function $f(x) = x^n,\ (x, n) \in \mathbb{N}$. Additionally, in section 4 exponential functions $\mathrm{Exp}(x),\ x \in \mathbb{N}$ representation is shown. In subsection (2.1) obtained results are applied to show finite difference of power.

Keywords. Power function, Binomial coefficient, Binomial Theorem, Finite difference, Perfect cube, Exponential function, Pascal's triangle, Series representation, Binomial Sum, Multinomial theorem, Multinomial coefficient, Binomial distribution, Hypercube

2010 Math. Subject Class. 30BXX
Personal website: kolosovpetro.github.io
ORCID: 0000-0002-6544-8880
e-mail: kolosov_94@mail.ua

Contents

1. Introduction and main results

In this paper particular pattern, that is sequence A287326 in OEIS, [**?**], which shows us necessary items to expand $x^3,\ x \in \mathbb{N}$ will be generalized and obtained results will be applied to show expansion of power function $f(x) = x^n,\ (x, n) \in \mathbb{N}$. In Section 3 received results are used to obtain finite differences of power function $f(x)$. Also, in this paper coefficient $\mathscr{U}(n, k)$ is introduced and its properties are shown.

Date: April 6, 2018.

Note that coefficient $\mathscr{U}(n,k)$ is k-th item of n-th row of triangle A287326. First, let review and basically describe Newtons Binomial Theorem, since our coefficient $\mathscr{U}(n,k)$ is derived from finite difference of perfect cubes, which is taken regarding Binomial expansion. In elementary algebra, the Binomial theorem describes the algebraic expansion of powers of a binomial. The theorem describes expanding of the power of $(x+y)^n$ into a sum involving terms of the form ax^by^c where the exponents b and c are nonnegative integers with $b+c=n$, and the coefficient a of each term is a specific positive integer depending on n and b. The coefficient a in the term of ax^by^c is known as the Binomial coefficient. The main properties of the Binomial Theorem are next

Properties 1.1. *Binomial Theorem properties*

(1) *The powers of x go down until it reaches $x_0 = 1$ starting value is n (the n in $(x+y)^n$)*
(2) *The powers of y go up from 0 ($y^0 = 1$) until it reaches n (also n in $(x+y)^n$)*
(3) *The n-th row of the Pascal's Triangle (see* [**?**], [**?**]*) will be the coefficients of the expanded binomial.*
(4) *For each line, the number of products (i.e. the sum of the coefficients) is equal to $x+1$*
(5) *For each line, the number of product groups is equal to 2^n*

According to the Binomial theorem, it is possible to expand any power of $x+y$ into a sum of the form (see [**?**], [**?**])

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \tag{1.2}$$

Let expand monomial x^n such that $(x,\ n) \in \mathbb{N}$ applying finite differences, that are reached by means of Binomial theorem (1.2)

Lemma 1.3. *Power function could be represented as discrete integral of its first order finite difference*

$$\begin{aligned} x^n &= \sum_{k=0}^{x-1} \Delta_h(x^n) \\ &= \sum_{k=0}^{x-1} \underbrace{nk^{n-1}h + \binom{n}{2} k^{n-2}h^2 + \cdots + \binom{n}{n-1} kh^{n-1} + h^n}_{\Delta_h[x^n]=(x+h)^n - x^n} \\ &= \sum_{j=0}^{x-1} \sum_{k=1}^{n} \binom{n}{k} j^{n-k} h^k,\ h \in \mathbb{R} \end{aligned} \tag{1.4}$$

Or, by means of Fundamental Theorem of Calculus

$$x^n = \int\limits_0^x nt^{n-1}dt = \sum_{k=0}^{x-1} \int_k^{k+1} nt^{n-1}dt = \sum_{k=0}^{x-1} (k+1)^n - k^n \tag{1.5}$$

Lemma 1.6. *From lemma (1.3) follows that finite difference of power $x^n,\ n \in \mathbb{N}$ could be reached by Binomial expansion of the form*

$$\Delta_h(x^n) = (x+h)^n - x^n = \sum_{k=1}^{n} \binom{n}{k} x^{n-k} h^k \tag{1.7}$$

Otherwise, let be a difference table of perfect cubes (see also [?], eq. 7)

(1.8)

x	x^3	$\Delta(x^3)$	$\Delta^2(x^3)$	$\Delta^3(x^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	6
6	216	127	42	6
7	343	169	48	6
8	512	217	54	
9	729	271		
10	1000			

Table 1: Difference table of $x^3,\ x \in \mathbb{N}$ up to third order, [?], eq. 7

Note that increment h is set to be $h = 1$ and $k > 2$-order difference is taken regarding to [?], [?]. Review Figure (1.8), then we can see that[1]

$$\begin{aligned} \Delta(0^3) &= 1 + 3! \cdot 0 \\ \Delta(1^3) &= 1 + 3! \cdot 0 + 3! \cdot 1 \\ \Delta(2^3) &= 1 + 3! \cdot 0 + 3! \cdot 1 + 3! \cdot 2 \\ \Delta(3^3) &= 1 + 3! \cdot 0 + 3! \cdot 1 + 3! \cdot 2 + 3! \cdot 3 \\ &\vdots \\ \Delta(x^3) &= 1 + 3! \cdot 0 + 3! \cdot 1 + 3! \cdot 2 + 3! \cdot 3 + \cdots + 3! \cdot x \end{aligned} \tag{1.9}$$

Obviously, the perfect cube x could written as

$$\begin{aligned} x^3 &= (1 + 3! \cdot 0) + (1 + 3! \cdot 0 + 3! \cdot 1) + (1 + 3! \cdot 0 + 3! \cdot 1 + 3! \cdot 2) + \cdots \\ &+ (1 + 3! \cdot 0 + 3! \cdot 1 + 3! \cdot 2 + \cdots + 3! \cdot (x-1)) \end{aligned} \tag{1.10}$$

Generalizing above expression, we have

$$\begin{aligned} x^3 &= x + (x-0) \cdot 3! \cdot 0 + (x-1) \cdot 3! \cdot 1 + (x-2) \cdot 3! \cdot 2 + \cdots \\ \cdots &+ (x-(x-1)) \cdot 3! \cdot (x-1) \end{aligned} \tag{1.11}$$

Provided that x is natural. Particularizing expression (1.11), one could have

$$x^3 = \sum_{m=0}^{x-1} 3! \cdot mx - 3! \cdot m^2 + 1 \tag{1.12}$$

Property 1.13. *Let be a sets $A(x) := \{1,\ 2,\ \ldots,\ x\}$, $B(x) := \{0,\ 1,\ \ldots,\ x\}$, $C(x) := \{0,\ 1,\ \ldots,\ x-1\}$ let be expression (1.12) written as*

$$T(x,\ C(x)) := \sum_{m \in C(x)} 3! \cdot mx - 3! \cdot m^2 + 1$$

where $x \in \mathbb{N}$ is variable and $U(x)$ is iteration set of (1.12), then we have equality

$$T(x,\ A(x)) = T(x,\ C(x)) \tag{1.14}$$

[1] The sequence A008458, [?] in OEIS, [?] is observed, the first order finite difference of consequent perfect cubes equals to $1 + a(n)$, where $a(n)$ is generating function of sequence A008458.

Let be expression (1.11) denoted as

$$U(x,\ C(x)) := x + 3! \cdot \sum_{m \in C(x)} mx - m^2$$

then

$$U(x,\ A(x)) = U(x,\ B(x)) = U(x,\ C(x)) \tag{1.15}$$

Other words, changing of iteration sets of(1.11) and (1.12) by $A(x),\ B(x),\ C(x)$ and $A(x),\ C(x)$, respectively doesn't change resulting value for each $x \in \mathbb{N}$.

Proof. Let be a plot $y(k) = 3! \cdot kx - 3! \cdot k^2 + 1,\ k \in \mathbb{R},\ 0 \leq k \leq 10$, given $x = 10$

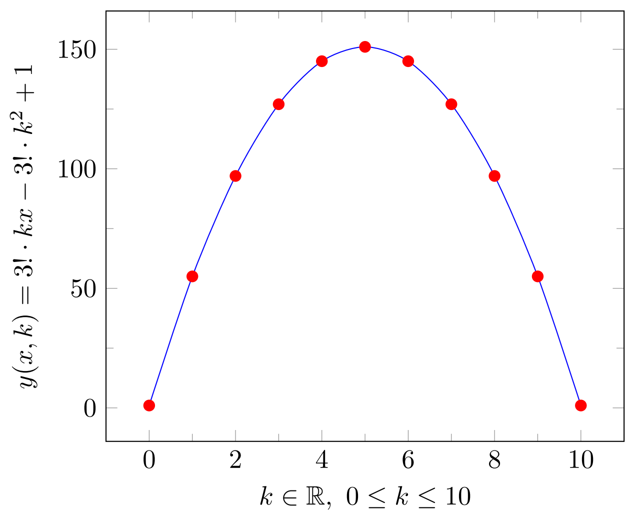

Figure 2. Plot of $y(k) = 3! \cdot kx - 3! \cdot k^2 + 1,\ k \in \mathbb{R},\ 0 \leq k \leq 10$, given $x = 10$

Obviously, being a parabolic function, it's symmetrical over $\frac{x}{2}$, hence equivalent $T(x,\ A(x)) = T(x,\ C(x)),\ x \in \mathbb{N}$ follows. Reviewing (1.11) and denote $u(t) = tx^{n-2} - t^2x^{n-3}$, we can conclude, that $u(0) = u(x)$, then equality of $U(x,\ A(x)) = U(x,\ B(x)) = U(x,\ C(x))$ immediately follows. This completes the proof. □

Review above property (1.13). Let be an example of triangle built using

$$y(n,k) = 3! \cdot kn - 3! \cdot k^2 + 1,\ 0 \leq k \leq n,\ (n,k) \in \mathbb{N} \tag{1.16}$$

over n from 0 to $n = 4$, where n denotes corresponding row and k shows the item of row n.

(1.17)

Row 0:					1				
Row 1:				1		1			
Row 2:			1		7		1		
Row 3:		1		13		13		1	
Row 4:	1		19		25		19		1

Figure 3. Triangle generated by (1.16) from 0 to $n = 4$, sequence A287326 in OEIS [?].

We can see that for each row according to variable $n = 0, 1, 2, 3, 4, ...,$ we have Binomial distribution of row items. One could compare Triangle (1.17) with Pascal's triangle [?], [?]

$$
\begin{array}{lccccccccc}
\text{Row 0:} & & & & & 1 & & & & \\
\text{Row 1:} & & & & 1 & & 1 & & & \\
\text{Row 2:} & & & 1 & & 2 & & 1 & & \\
\text{Row 3:} & & 1 & & 3 & & 3 & & 1 & \\
\text{Row 4:} & 1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

Figure 4. Pascal's triangle up to forth row, sequence A007318 in OEIS, [?], [?].

Note that n-th row sum of Triangle (1.17) returns us perfect cube n^3. Hereby, the follow question is stated

Question 1.18. *Has the Triangle (1.17) any connection with Pascal's Triangle or others like Stirling or Euler, and is it exist similar patterns in order to receive expansion of x^j $j > 3$?*

2. Answer to the Question (1.18)

In section particular pattern (1.17), that is sequence A287326 in OEIS, [?], which shows us necessary items to expand $x^3, \ x \in \mathbb{N}$ will be generalized and obtained results will be applied to show expansion of power function $f(x) = x^n, \ (x, n) \in \mathbb{N}$. To answer to the question (1.18), let review our Triangle (1.17) again. Next, let take away from each item k, such that $0 < k < n$ of Triangle (1.17) the value of n^2, then we have

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 3 & & 1 & & \\
 & 1 & & 4 & & 4 & & 1 & \\
1 & & 3 & & 9 & & 3 & & 1
\end{array} \tag{2.1}
$$

Figure 5. Triangle generated by $\begin{cases} 3! \cdot kn - 3! \cdot k^2 + 1 - n^2, \ 0 < k < n \\ 1, \ k \in \{0, \ n\} \end{cases}$

We can observe that summation of n-th row of Triangle (2.1) over k from 0 to $n-1$ returns us the n^2. It's very easy to see that removing n^1 from each item k, such that $0 < k < n$ of Triangle (2.1) and summing up received n-th rows over k from 0 to $n-1$ will result n^1, let show it

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 1 & & 1 & & \\
 & 1 & & 1 & & 1 & & 1 & \\
1 & & -1 & & 5 & & -1 & & 1
\end{array} \tag{2.2}
$$

Figure 6. Triangle generated by $\begin{cases} 3!\cdot kn - 3!\cdot k^2 + 1 - n^2 - n^1, \ 0<k<n \\ 1, \ k\in\{0, \ n\} \end{cases}$

Review the Triangle (2.2), we can say that above statement holds. Reviewing our Triangles (2.1), (2.2), let define generalized item $V_M(n, \ k)$

Definition 2.3.

$$V_M(n, \ k) := \begin{cases} n^0 + n^1 + \cdots + n^M, \ 0<k<n \\ 1, \ k\in\{0, \ n\} \end{cases} \tag{2.4}$$

Property 2.5. *From definition (2.3) follows the equality between items $V_M(n, \ k)$ in range $k \in \{1, \ n-1\}$*

$$\begin{aligned} \forall k\in\{1, \ n-1\}: \ V_M(n, \ k) &= V_M(n, \ k+1 \leq n-1) \\ &= V_M(n, \ k+2 \leq n-1) \\ &\cdots \\ &= V_M(n, \ k+j \leq n-1), \ j\in\mathbb{N} \end{aligned} \tag{2.6}$$

Note that k-th items of Triangles (2.1), (2.2), such that $0<k<n$ are $V_1(n, \ k), \ V_2(n, \ k)$, respectively. Reviewing our Triangles (2.1), (2.2), we could observe the identity

$$n^M = \sum_{k=0}^{n-1} V_{M-1}(n, \ k), \qquad M\in\{1,2,3\} \tag{2.7}$$

Example 2.8. Review (2.7), let be $n=4, \ M=3$, then

$$\begin{aligned} 4^3 &= V_2(4, \ 0) + V_2(4, \ 1) + V_2(4, \ 2) + V_2(4, \ 3) \\ &= 1 + \underbrace{1+4+4^2}_{V_2(4, \ 1)} + \underbrace{1+4+4^2}_{V_2(4, \ 2)} + \underbrace{1+4+4^2}_{V_2(4, \ 3)} \\ &= 1 + 3(4^0+4^1+4^2) = 1+21+21+21 \end{aligned}$$

Hereby, let be Theorem

Theorem 2.9. *Each power function $f(x)=x^n$ such that $(x,n)\in\mathbb{N}$ could be expanded next way*

$$x^n = \sum_{k=0}^{x-1} V_{n-1}(x, \ k) \tag{2.10}$$

Proof. Recall Triangle, consisting of items $V_0(x, \ k)$, that is analog of (2.2)

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 1 & & 1 & & \\ & 1 & & 1 & & 1 & & 1 & \\ 1 & & 1 & & 1 & & 1 & & 1 \end{array} \tag{2.11}$$

Figure 7. Triangle generated by $V_0(x, \ k)$, sequence A000012 in OEIS, [?].

Obviously, summation over rows of triangle (2.11) from 0 to $x-1$ gives us the x^1. Let show a second power by means of $V_0(x,\ k)$, we have accordingly

$$x^2 = 1 + \underbrace{(V_0(x,\ 1)+x)}_{V_1(x,\ 1)} + \underbrace{(V_0(x,\ 2)+x)}_{V_1(x,\ 2)} + \cdots + \underbrace{(V_0(x,\ x-1)+x)}_{V_1(x,\ x-1)} \tag{2.12}$$

For example, consider the sum of third row of triangle (2.11), then we receive 3^1. Hence, by power property

$$x^n = \sum_{k=0}^{x-1} x^{n-1} \tag{2.13}$$

we have to add twice by 3^1 to receive 3^2, i.e

$$3^2 = 1 + (3+1) + (3+1)$$

Generalizing above result, for each $(x,n) \in \mathbb{N}$ we have identity

$$\begin{aligned} \sum_{k=0}^{x-1} V_{n-1}(x,\ k) &= \\ &= 1 + \underbrace{(x^0+x^1+\cdots+x^{n-1}) + \cdots + (x^0+x^1+\cdots+x^{n-1})}_{x-1 \text{ times}} \\ &= 1 + (x-1)(x^0+x^1+x^2+\cdots+x^{n-1}) \\ &= x + (x-1)x + (x-1)x^2 + (x-1)x^3 + \cdots + (x-1)x^{n-1} \\ &= 1 + (x-1)V_{n-1}(x,\ k) \\ &= 1 + x^n - x^0 = x^n \end{aligned} \tag{2.14}$$

This completes the proof. □

Also, (2.14) could be rewritten as

$$\begin{aligned} x^n &= 1 + \underbrace{(x-1) + (x-1)x + \cdots + (x-1)x^{n-1}}_{n \text{ times}} \\ &= 1 + \sum_{k=0}^{n-1} (x-1)x^k \\ &= 1 + \sum_{k=0}^{n-1} x^{k+1} - x^k,\ \forall (x,\ n) \in \mathbb{N} \end{aligned} \tag{2.15}$$

Define the power function $f(x) = x^n$, such that $(x,\ n) \in \mathbb{N}$ and exponential function $g(x,\ k) = x^k,\ x \in \mathbb{N}$, then expression (2.15) shows us the relation between exponential and power functions with natural base and exponent

$$f(x) = 1 + \sum_{k=0}^{n-1} g(x,\ k+1) - g(x,\ k) \tag{2.16}$$

Reader could also notice the connection between Maclaurin expansion $\frac{1}{1-x} = x^0 + x^1 + x^2 + x^3 + \cdots,\ -1 < x < 1$ and (2.14), that is

$$\frac{x^n}{x-1} - \underbrace{\frac{1}{x-1}}_{-\frac{1}{1-x}} = x^0 + x^1 + x^2 + \cdots + x^{n-1},\ \forall (x,\ n) \in \mathbb{N} \tag{2.17}$$

Next, let review and apply our results on Binomial Related expansion of monomial $f(x) = x^n,\ (x,\ n) \in \mathbb{N}$. Recall (1.4)

$$\begin{aligned} x^n &= \sum_{k=0}^{x-1} \Delta_h[x^n] \\ &= \sum_{k=0}^{x-1} \underbrace{nk^{n-1} + \binom{n}{2}k^{n-2} + \cdots + \binom{n}{n-1}k + 1}_{\Delta_h[x^n]=(x+h)^n - x^n} \\ &= \sum_{j=0}^{x-1}\sum_{k=1}^{n} \binom{n}{k} j^{n-k},\ x \in \mathbb{N} \end{aligned} \tag{2.18}$$

Then, it follows for each $(x, m, n) \in \mathbb{N}$

$$\begin{aligned} x^{n+1} &= \sum_{j=0}^{x-1}\sum_{k=1}^{n+1} \binom{n}{2} j^{n+1-k} = \sum_{j=0}^{x-1}\sum_{k=1}^{n} \binom{n}{2} j^{n-k} + x^n \\ x^{n+m} &= \sum_{j=0}^{x-1}\sum_{k=1}^{n} \binom{n}{2} j^{n-k} + x^n + x^{n+1} + \cdots + x^{n+m-1},\ x \in \mathbb{N} \end{aligned} \tag{2.19}$$

To show one more property of $V_M(n,\ k)$, let build Triangle given $V_2(n,\ k)$

$$\begin{array}{ccccccccc} &&&& 1 &&&& \\ &&& 1 && 1 &&& \\ && 1 && 3 && 1 && \\ & 1 && 4 && 4 && 1 & \\ 1 && 5 && 5 && 5 && 1 \end{array} \tag{2.20}$$

Figure 8. Triangle generated by $V_2(n,\ k)$ over n from 0 to 9.

Reviewing above triangle, we could observe that summation of intermediate column gives us well known identity,

$$x^2 = \sum_{k=0}^{|x|-1} 2k + 1,\ x \in \mathbb{Z} \tag{2.21}$$

Applying $V_2(n,\ k)$, we receive analog of above identity

$$x^2 = \sum_{k=0}^{n-1} V_2(2k,\ k),\ \forall(x,\ n) \in \mathbb{N} \tag{2.22}$$

Note that upper expression (2.22) is partial case of (2.19), when $n + m = 2$. Recall the Binomial $(x + y)^n$, by means of (2.14) we have expansion

$$(x + y)^n = 1 + (x + y - 1)V_{n-1}(x + y,\ k) \tag{2.23}$$

Hereby, let be lemma

Lemma 2.24. *Relation between binomial expansion and* $V_{n-1}(x,\ k)$

$$
\begin{aligned}
(2.25)(x+y)^n &= \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k = 1+(x+y-1)V_{n-1}(x+y,\ k) \\
&= 1+(x+y-1)(x+y)^0+(x+y-1)(x+y)^1+\cdots \\
&+ +(x+y-1)(x+y)^{n-1} \\
&= 1+x((x+y)^0+(x+y)^1+(x+y)^2+\cdots+(x+y)^{n-1}) \\
&+ y((x+y)^0+(x+y)^1+(x+y)^2+\cdots+(x+y)^{n-1}) \\
&- ((x+y)^0+(x+y)^1+(x+y)^2+\cdots+(x+y)^{n-1})
\end{aligned}
$$

Multinomial case could be built as well as Binomial, hereby

$$
\begin{aligned}
(x_1+x_2+\cdots+x_k) &= \\
(2.26) \qquad &= 1+(x_1+x_2+\cdots+x_k-1)(x_1+x_2+\cdots+x_k)^0 \\
&+ (x_1+x_2+\cdots+x_k-1)(x_1+x_2+\cdots+x_k)^1 \\
&+ (x_1+x_2+\cdots+x_k-1)(x_1+x_2+\cdots+x_k)^2 \\
&\vdots \\
&+ (x_1+x_2+\cdots+x_k-1)(x_1+x_2+\cdots+x_k)^{n-1}
\end{aligned}
$$

2.1. **Finite Differences.** In this subsection let apply received in previous section results to show finite differences of power function $f(x)=x^n$, such that $(x,\ n)\in\mathbb{N}$. From (2.14) we know identity

$$f(x)=1+(x-1)(x^0+x^1+x^2+\cdots+x^{n-1})=1+(x-1)V_{n-1}(x,\ k)=x^n \tag{2.27}$$

Then, its finite difference $\Delta f(x)$ suppose to be

$$
\begin{aligned}
(2.28) \qquad \Delta f(x) &= f(x+1)-f(x)= \\
&= [1+\underbrace{x((x+1)^0+(x+1)^1+\cdots+(x+1)^{n-1})}_{xV_{n-1}(x+1,\ k)}] \\
&- [1+\underbrace{(x-1)(x^0+x^1+x^2+\cdots+x^{n-1})}_{(x-1)V_{n-1}(x,\ k)=x^n-1}] \\
&= xV_{n-1}(x+1,\ k)-(x-1)V_{n-1}(x,\ k) \\
&= xV_{n-1}(x+1,\ k)-x^n-1 \\
&= xV_{n-1}(x+1,\ k)-xV_{n-1}(x,\ k)+V_{n-1}(x,\ k) \\
&= x[V_{n-1}(x+1,\ k)-V_{n-1}(x,\ k)]+V_{n-1}(x,\ k)
\end{aligned}
$$

Example 2.29. Consider the example for $f(x)=x^n,\ x=3,\ n=3$, then applying (2.27), we have

$$
\begin{aligned}
(2.30) \qquad \Delta f(3) &= f(4)-f(3) \\
&= [1+3((3+1)^0+(3+1)^1+(3+1)^2)] \\
&- [1+(3-1)(3^0+3^1+3^2)] \\
&= [3((3+1)^0+(3+1)^1+(3+1)^2)] \\
&- [(3-1)(3^0+3^1+3^2)] \\
&= 63-26=37
\end{aligned}
$$

Let generalize (2.27) and show high order finite difference of power $f(x) = x^n$ by means of $V_{n-1}(x,\ k)$, that is

$$\Delta^m(x^n) = \sum_{k=0}^{m-1}(x-k)[V_{n-1}(x+m-k,t) - V_{n-1}(x+m-k+1,t)], \tag{2.31}$$

where $t \neq 0$. Derivative of $f(x) = x^n$ could be written regarding to

$$\frac{df(x)}{dx} = \lim_{h\to 0}\left[\frac{xV_{n-1}(x+h,\ k) - (x-h)V_{n-1}(x,\ k)}{h}\right] \tag{2.32}$$

3. Properties of Triangle (1.17) and other expansions

Review the triangle (1.17), define the k-th, $0 \leq k \leq n$, item of n-th row of triangle as

Definition 3.1.

$$\mathscr{U}(n,k) := 3! \cdot nk - 3! \cdot k^2 + 1,\ 0 \leq k \leq n \tag{3.2}$$

Note that definition (3.1) also could be rewritten as

$$\mathscr{U}(n,k) = 3! \cdot nk - 3! \cdot n^0 k^2 + n^0,\ 0 \leq k \leq n \tag{3.3}$$

Let us approach to show a few properties of triangle (1.17)

Properties 3.4. *Properties of triangle (1.17).*

(1) *Summation of items $\mathscr{U}(n,k)$ of n-th row of triangle (1.17) over k from 0 to $n-1$ returns perfect cube n^3 as binomial of the form*

$$\sum_{k=0}^{n-1}\mathscr{U}(n,k) = A_{0,n}n - B_{0,n} = n^3, \tag{3.5}$$

Since the property (1.14) holds, (3.5) could be rewritten as

$$\sum_{k=1}^{n}\mathscr{U}(n,k) = A_{1,n}n - B_{1,n} = n^3, \tag{3.6}$$

where $A_{\overline{0,1},n}$ and $B_{\overline{0,1},n}$ - integers ***depending*** *on variable $n \in \mathbb{N}$ and on sets $U(n),\ S(n)$, respectively.*

(2) *Recurrence relation between $A_{0,n}$ and $A_{1,n}$*

$$A_{0,n+1} = A_{1,n},\ n \geq 1$$

(3) *Induction by power. Summation of items of n-th row of triangle (1.17), multiplied by n^{m-3}, from 0 to $n-1$ returns n^m*

(4) *Summation of items $\mathscr{U}(n,k)$ of n-th row of triangle (1.17) over k from 0 to n returns $n^3 + n^0$*

$$\sum_{k=0}^{n}\mathscr{U}(n,k) = n^3 + 1 \tag{3.7}$$

(5) *Induction by power. Summation of each n-th row of triangle (1.17) multiplied by n^{m-3} from 0 to n returns $n^m + n^{m-3}$*

(6) *First item of each row's number corresponding to central polygonal numbers sequence a(n) (sequence A000124 in OEIS,* [?] *returns finite difference* $\Delta[n^3]$ *of consequent perfect cubes. For example, let be the k-th row of triangle (1.17), such that k is central polygonal number, i.e* $k = \frac{n^2+n+2}{2}$, $n = 0,\ 1,\ 2, ...,\ \mathbb{N}$, *then item*

$$\mathscr{U}\left(\frac{n^2+n+2}{2}, 1\right) = \Delta_h(n^3),\ h = 1 \tag{3.8}$$

(7) *Items of (1.17) have Binomial distribution over rows*

(8) *The linear recurrence, for any k and* $n > 0$

$$2\mathscr{U}(n,k) = \mathscr{U}(n+1,k) + \mathscr{U}(n-1,k) \tag{3.9}$$

(9) *Linear recurrence, for each* $n > k$

$$2\mathscr{U}(n,k) = \mathscr{U}(2n-k,k) + \mathscr{U}(2n-k,0) \tag{3.10}$$

(10) *From (3.8) follows that*

$$n^3 = \sum_{k=0}^{n-1} \mathscr{U}(n,k) = \sum_{k=0}^{n-1} \mathscr{U}\left(\frac{n^2+n+2}{2}, 1\right) \tag{3.11}$$

(11) *Triangle is symmetric, i.e*

$$\mathscr{U}(n,k) = \mathscr{U}(n,n-k)$$

(12) *Relation between Rascal Triangle A077028 and Triangle (1.17) A287326*
*A287326(n,k) = 6 * A077028(n,k) - 5*

In (3.5) is noticed, that summation of each n-th row of Triangle (1.17) from 0 to $n-1$ returns perfect cube n^3, then, by properties (3.8), (3.9), (3.10), for each given number $x \in \mathbb{N}$ the x^n could be easy found via multiplication of each term of (3.5) by x^{n-3}

$$\begin{aligned} x^n &= \sum_{k=0}^{x-1} \mathscr{U}(x,k)x^{n-3} = \frac{1}{2}\sum_{k=0}^{x-1} \left[\mathscr{U}(x+1,k) + \mathscr{U}(x-1,k)\right] x^{n-3} \\ &= \sum_{k=0}^{x-1} \frac{1}{2}\left[\mathscr{U}(2x-k,k) + \mathscr{U}(2x-k,0)\right] x^{n-3} \\ &= \sum_{k=0}^{x-1} \frac{1}{2}\mathscr{U}\left(\frac{x^2+x+2}{2}, 1\right) x^{n-3} \\ &= \sum_{k=0}^{x-1} \frac{1}{2}\left[\mathscr{U}\left(\frac{x^2+x}{2}, 1\right) + \mathscr{U}\left(\frac{x^2+x+4}{2}, 1\right)\right] x^{n-3} \\ &= \sum_{k=0}^{x-1} \frac{1}{2}\left[\mathscr{U}\left(\binom{n+1}{2}, 1\right) + \mathscr{U}\left(\binom{n+1}{2} + \binom{2}{1}, 1\right)\right] x^{n-3} \end{aligned} \tag{3.12}$$

To show other way of representation of power, let move the x from (1.12), $x + 3!\sum mx - m^2$, under the sum operator and change iteration set from $\{0,\ x-1\}$ to $\{1,\ x-1\}$, then we get

$$x^3 = \sum_{m=1}^{x-1} 3! \cdot mx - 3! \cdot m^2 + \frac{x}{(x-1)}, \qquad x \neq 1,\ x \in \mathbb{N} \tag{3.13}$$

Review right part of (3.13), let be item $\frac{x}{x-1}$ written as $\frac{x}{x-1} = \frac{x+1-1}{x-1} = 1 + \frac{1}{x-1}$, given the power $n > 3$, multiplying each term of (3.13) by x^{n-3} we can observe that

$$x^n - 1 = \sum_{m=1}^{x-1} \mathscr{U}(x,m)x^{n-3} + x^{n-4} + x^{n-5} + \cdots + x + 1 \tag{3.14}$$

Applying properties (3.8), (3.9), (3.10), we can rewrite (3.14) as

$$\begin{aligned} x^n - 1 &= \sum_{k=1}^{x-1} \frac{1}{2}\left[\mathscr{U}(2x-k,k) + \mathscr{U}(2x-k,0)\right] x^{n-3} + x^{n-4} + \cdots + x + 1 \\ &= \sum_{k=1}^{x-1} \frac{1}{2}\left[\mathscr{U}(x+1,k) + \mathscr{U}(x-1,k)\right] x^{n-3} + x^{n-4} + \cdots + x + 1 \qquad (3.15) \\ &= \sum_{m=0}^{x-1} \frac{1}{2}\left[\mathscr{U}\left(\tfrac{x^2+x}{2},1\right) + \mathscr{U}\left(\tfrac{x^2+x+4}{2},1\right)\right] x^{n-3} + x^{n-4} + \cdots + x + 1 \\ &= \sum_{k=1}^{x-1} \mathscr{U}\left(\tfrac{x^2+x+2}{2},1\right) x^{n-3} + x^{n-4} + \cdots + 1 \end{aligned}$$

Moving 1 from left part of (3.14) under sum operator, we add a term $\frac{1}{x-1}$ to initial function $\mathscr{U}(x,m)x^{n-3} + x^{n-4} + x^{n-5} + \cdots + x + 1$. By means of expansion $\frac{1}{1-x} = -\frac{1}{x-1} = 1 + x + x^2 + x^3 + \cdots$, the (3.14) could be rewritten accordingly

$$x^n = \sum_{m=1}^{x-1} \mathscr{U}(x,m)x^{n-3} + x^{n-4} + \cdots + x + 1 - 1 - x^2 - x^3 - \cdots \tag{3.16}$$

Generalizing (3.16) we have

$$\begin{aligned} x^n &= \sum_{m=1}^{x-1} \left[\mathscr{U}(x,m) - 1\right] x^{n-3} - x^{n-2} - x^{n-1} - \cdots \qquad (3.17) \\ &= \sum_{m=1}^{x-1} \left[\mathscr{U}\left(\frac{x^2+x+2}{2},1\right) - 1\right] x^{n-3} - x^{n-2} - x^{n-1} - \cdots \end{aligned}$$

3.1. **Generalized Binomial Series by means of properties (3.5), (3.6).** Reviewing properties (3.5), (3.6) we can say that for each $x = x_0 \in \mathbb{N}$ holds

$$x^n = A_{\overline{0,1},x} x^{n-2} - B_{\overline{0,1},x} x^{n-3} \tag{3.18}$$

Rewrite the right part of (3.18) regarding to itself as recursion

$$\begin{aligned} x^n &= A_{\overline{0,1},x}(A_{\overline{0,1},x}x^{n-4} - B_{\overline{0,1},x}x^{n-5}) - B_{\overline{0,1},x}(A_{\overline{0,1},x}x^{n-5} - B_{\overline{0,1},x}x^{n-6}) \\ &= A^2_{\overline{0,1},x}x^{n-4} - 2A_{\overline{0,1},x}B_{\overline{0,1},x}x^{n-5} + B^2_{\overline{0,1},x}x^{n-6} \qquad (3.19) \end{aligned}$$

Reviewing above expression we can observe Binomial coefficients before each $A_{\overline{0,1},x} \cdot B_{\overline{0,1},x}$. Continuous j-times recursion of right part of (3.18) gives us

$$x^n = \sum_{k=0}^{j} (-1)^k \binom{j}{k} A^{j-k}_{\overline{0,1},x} B^k_{\overline{0,1},x} x^{n-2j-k} \tag{3.20}$$

Suppose that we want to repeat action (3.19) infinite-many times, then

$$x^n = \binom{j}{0} A^{j}_{\overline{0,1},x} B^{0}_{\overline{0,1},x} x^{n-2j} - \binom{j}{1} A^{j-1}_{\overline{0,1},x} B^{1}_{\overline{0,1},x} x^{n-2j-1} + \cdots - \cdots + \sum_{k}^{\infty} (-1)^k \binom{j}{k} A^{j-k}_{\overline{0,1},x} B^{k}_{\overline{0,1},x} x^{n-2j-k} + \cdots \tag{3.21}$$

We know the solutions of above equation (3.18) for all $A_{\overline{0,1},x}$, $B_{\overline{0,1},x}$ that present in follow table. The table arranged next way

$x =$	$A_{0,x} =$	$B_{0,x} =$	$A_{1,x} =$	$B_{1,x} =$
1	1	0	6	5
2	6	4	18	28
3	18	25	36	81
4	36	80	60	176
5	60	175	90	325
6	90	324	126	540
7	126	539	168	833
8	168	832	216	1216
9	216	1215	270	1701
10	270	1700	330	2300

Table 9. Array of coefficients $A_{\overline{0,1},x}$, $B_{\overline{0,1},x}$ over x from 1 to 10.

Sequence $B_{1,x}$ is generated by $2x^3+3x^2,\ x\in\mathbb{N}$, sequence A275709 in OEIS, [?]. Sequence $A_{1,x}$ is generated by $3x^2+3x,\ x\in\mathbb{N}$, sequence A028896 in OEIS, [?].

4. Exponential Function Representation

Since the exponential function $f(x)=e^x,\ x\in\mathbb{N}$ is defined as infinite summation of $\frac{x^n}{n!},\ n=0,\ 1,\ 2,\ ...,\ \infty$ over n (see [?]). Then (3.12) could be applied, hereby

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=1}^{x-1} \left[\mathscr{U}(x,m)-1\right] x^{n-3} - x^{n-2} - x^{n-1} - \cdots \tag{4.1}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=1}^{x-1} \left[\mathscr{U}\left(\frac{x^2+x+2}{2},1\right)-1\right] x^{n-3} - x^{n-2} - x^{n-1} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=1}^{x-1} \left[\frac{1}{2}\left[\mathscr{U}(x+1,k)+\mathscr{U}(x-1,k)\right]-1\right] x^{n-3} - x^{n-2} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=1}^{x-1} \left[\frac{1}{2}\left[\mathscr{U}(2x-k,k)+\mathscr{U}(2x-k,0)\right]-1\right] x^{n-3} - x^{n-2} - \cdots$$

From (4.1) for each $x \in \mathbb{N}$ we get

$$e^x - e = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=1}^{x-1} \mathscr{U}(x,m) x^{n-3} + x^{n-4} + \cdots + x + 1 \tag{4.2}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=1}^{x-1} \frac{1}{2} \left[\mathscr{U}(x+1,m) + \mathscr{U}(x-1,m)\right] x^{n-3} + x^{n-4} + \cdots + x + 1$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=1}^{x-1} \mathscr{U}\left(\frac{x^2+x+2}{2}, 1\right) x^{n-3} + x^{n-4} + \cdots + x + 1$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{m=1}^{x-1} \tfrac{1}{2} \left[\mathscr{U}\left(\tfrac{x^2+x}{2}, 1\right) + \mathscr{U}\left(\tfrac{x^2+x+4}{2}, 1\right)\right] x^{n-3} + x^{n-4} + \cdots + 1$$

5. Relation between Pascal's Triangle and Volume of Hypercubes

In this section let review and generalize well known fact about connection between row sums of Pascal triangle and 2−dimension Hypercube, recall property

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \tag{5.1}$$

Now, let multiply each k-th term of of n-th row of Pascal's triangle [?] by 2^k

1

1 2

1 4 4

1 6 12 8

1 8 24 32 16

Figure 10. Triangle built by $\binom{n}{k} \cdot 2^k, \; 0 \le k \le n \le 4$.

We can notice that

$$\sum_{k=0}^{n} \binom{n}{k} \cdot 2^k = 3^n, \quad 0 \le k \le n, \quad (n, \; k) \in \mathbb{N} \tag{5.2}$$

Hereby, let be theorem

Theorem 5.3. *Volume of n-dimension hypercube with length m could be calculated as*

$$m^n = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} (-1)^{k-j} m^j \tag{5.4}$$

where m and n - positive integers.

Proof. Recall induction over m, in (5.1) is shown a well-known example for $m = 2$.

$$2^n = \sum_{k=0}^{n} \binom{n}{k} (2-1)^k \tag{5.5}$$

Review (5.5) and suppose that

$$(\underbrace{2+1}_{m=3})^n=\sum_{k=0}^{n}\binom{n}{k}\cdot(\underbrace{(2-1)+1}_{m-1})^k \tag{5.6}$$

And, obviously, this statement holds by means of Newton's Binomial Theorem [?], [?] given $m=3$, more detailed, recall expansion for $(x+1)^n$ to show it.

$$(x+1)^n=\sum_{k=0}^{n}\binom{n}{k}x^k \tag{5.7}$$

Substituting $x=2$ to (5.7) we have reached (5.6).
Next, let show example for each $m\in\mathbb{N}$. Recall Binomial theorem to show this

$$m^n=\sum_{k=0}^{n}\binom{n}{k}\cdot(m-1)^k \tag{5.8}$$

Hereby, for $m+1$ we receive Binomial theorem again

$$(m+1)^n=\sum_{k=0}^{n}\binom{n}{k}\cdot m^k \tag{5.9}$$

Review result from (5.8) and substituting Binomial expansion $\sum_{j=0}^{k}\binom{k}{j}(-1)^{n-k}m^j$ instead $(m-1)^k$ we receive desired result

$$\begin{aligned} m^n &= \sum_{k=0}^{n}\binom{n}{k}\cdot\underbrace{(m-1)^k}_{\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}m^j}=\sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}m^j \\ &= \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}\binom{k}{j}(-1)^{k-j}m^j \end{aligned} \tag{5.10}$$

This completes the proof. □

The (5.5) is analog of MacMillan Double Binomial Sum (see equation 13 in [?]).

Lemma 5.11. *Number of elements $k-$face elements $\mathscr{E}_k(\mathbf{Y}_n^p)$ of Generalized Hypercube $\mathbf{Y}_n^p$ equals to*

$$\mathscr{E}_k(\mathbf{Y}_n^p)=\sum_{j=0}^{k}\binom{n}{k}\binom{k}{j}(-1)^{k-j}(p-1)^j \tag{5.12}$$

We also can observe that

$$\begin{aligned} x^n-1 &= \sum_{k=1}^{n}\binom{n}{k}+x^{n-1}=\sum_{k=1}^{n}\left(\binom{n}{k}+\sum_{k=1}^{n}\left[\binom{n}{k}+x^{n-2}\right]\right) \\ &= \sum_{k=1}^{n}\left(\underbrace{\binom{n}{k}+\sum_{k=1}^{n}\left(\binom{n}{k}+\sum_{k=1}^{n}\left(\binom{n}{k}+\cdots\right)\right)}_{n-\text{times continued summation}}\right) \end{aligned} \tag{5.13}$$

In sense of Gauss Continued fraction operator $\mathbf{K}$, we can receive

$$x^n - 1 = \sum_{k=1}^{n} \left[\binom{n}{k} + \underset{j=1}{\overset{n}{\mathrm{K}}} \left\{ \sum_{k=1}^{n-j} \binom{n}{k} + x^{n-j} \right\} \right]$$

6. Conclusion and future research

In this paper particular pattern, that is sequence A287326 in OEIS, [**?**], which shows us necessary items to expand $x^3,\ x \in \mathbb{N}$ will be generalized and obtained results will be applied to show expansion of power function $f(x) = x^n,\ (x,n) \in \mathbb{N}$. The coefficient $\mathscr{U}(n,k)$ was introduced in definition (3.1), its properties are shown in (3.4). Power function's $x^n,\ (x,n) \in \mathbb{N}$ expansion firstly shown in (2.15) and other versions are shown below. In section 4 we show a various representations of exponential function $e^x,\ x \in \mathbb{N}$. We attach a Wolfram Mathematica codes of most important equations in Application 1. Extended version of Triangle (1.17) is attached in Application 2. Future research should be done in $\mathscr{U}(n^m, k^j),\ m,j \in \mathbb{N}$ to verify its properties. A research on finding difference and consequently derivative using (3.12) and extend it over real functions applying Taylor's theorem also could be done. In subsection (2.1) finite differences by means of $V_M(n,k)$ are shown. The difference of received method from Binomial theorem lays in fact that for Binomial expansion n-th row of Pascal's triangle denotes the power of x, but in case of Triangle (1.17) n-th row denotes variable to power, corresponding to $V_M(n,k)$. Reviewing MacMillan Double Binomial Sum [**?**], we can observe that it reached by means of Stirling Numbers of Second Kind.

7. Application 1. Wolfram Mathematica codes of some expressions

In this section Wolfram Mathematica codes of most expressions are shown. Note that Mathematica `.cdf`-file of all mentioned expressions is available for download at this link. The `.txt`-file reader could find here. Define coefficient $\mathscr{U}(n,k)$, definition (3.1)

```
U[n_, k_] := 3!*n*k − 3!*k^2 + 1
```

Check of property (3.8)

```
U[(n^2 + n + 2)/2, 1]
```

Check of expression (3.12), x^n

```
1/2*Sum[(U[2*x − k, k] + U[2*x − k, 0])*x^(n − 3), {k, 0, x − 1}]

1/2*Sum[(U[x + 1, k] + U[x − 1, k])*x^(n − 3), {k, 0, x − 1}]

1/2*Sum[(U[(k^2 + k)/2, 1] + U[(k^2 + k + 4)/2, 1])*x^(n − 3),
        {k, 0, x − 1}]
```

```
Sum[U[x, k]*x^(n-3), {k, 0, x - 1}]

Sum[U[(k^2 + k + 2)/2, 1]*x^(n - 3), {k, 0, x - 1}]
```

Generating formula of Triangle (1.17), Figure 3

```
Column[Table[U[n, k], {n, 0, 5}, {k, 0, n}], Center]
```

Expression (3.14), $x^n - 1$

```
Sum[U[x, m]*x^(n - 3) + Sum[x^(n - t), {t, 4, n}], {m, 1, x - 1}]
```

Expression (3.15), $x^n - 1$ using properties (3.8), (3.9), (3.10)

```
Sum[1/2*(U[2*x - k, k] + U[2*x - k, 0])*x^(n - 3) +
    Sum[x^(n - t), {t, 4, n}], {k, 1, x - 1}]

    Sum[1/2*(U[x + 1, k] + U[x - 1, k])*x^(n - 3) +
    Sum[x^(n - t), {t, 4, n}], {k, 1, x - 1}]

    Sum[1/2*(U[(k^2 + k)/2, 1] + U[(k^2 + k + 4)/2, 1])*x^(n - 3) +
    Sum[x^(n - t), {t, 4, n}], {k, 1, x - 1}]

    Sum[U[(k^2 + k + 2)/2, 1]*x^(n - 3) + Sum[x^(n - t), {t, 4, n}],
    {k, 1, x - 1}]
```

Expression (3.16), x^n

```
Sum[U[x, m]*x^(n - 3) + Sum[x^(n - t), {t, 4, n}] -
    Sum[x^j, {j, 0, Infinity}], {m, 1, x - 1}]
```

Expression (3.17), generalized version of (3.16), x^n

```
Sum[(U[x, m] - 1)*x^(n - 3) -
    Sum[x^(n - 3 + j), {j, 1, Infinity}], {m, 1, x - 1}]

    Sum[(U[(m^2 + m + 2)/2, 1] - 1)*x^(n - 3) -
    Sum[x^(n - 3 + j), {j, 1, Infinity}], {m, 1, x - 1}]
```

Section 4, Expression (4.1), e^x representation

```
Sum[1/n!*Sum[(U[x, m] − 1)*x^(n − 3) − Sum[x^(n − 3 + j),
        {j, 1, Infinity}], {m, 1, x − 1}], {n, 0, Infinity}]

        Sum[1/n!*Sum[(U[(m^2 + m + 2)/2, 1] − 1)*x^(n − 3) −
        Sum[x^(n − 3 + j), {j, 1, Infinity}], {m, 1, x − 1}],
        {n, 0, Infinity}]

        Sum[1/n!*Sum[(1/2*(U[x + 1, m] + U[x − 1, m]) − 1)*x^(n − 3) −
        Sum[x^(n − 3 + j), {j, 1, Infinity}], {m, 1, x − 1}],
        {n, 0, Infinity}]

        Sum[1/n!*Sum[(1/2*(U[2x−m, m] + U[2x−m, 0]) − 1)*x^(n − 3) −
        Sum[x^(n − 3 + j), {j, 1, Infinity}], {m, 1, x − 1}],
        {n, 0, Infinity}]
```

Section 4, Expression (4.2), $e^x - e$ representation

```
Sum[1/n!*Sum[U[x, m]*x^(n − 3) +
        Sum[x^(n − t), {t, 4, n}], {m, 1, x − 1}], {n, 0, Infinity}]

        Sum[1/n!*Sum[1/2*(U[x + 1, m] + U[x − 1, m])*x^(n − 3) +
        Sum[x^(n − t), {t, 4, n}], {m, 1, x − 1}], {n, 0, Infinity}]

        Sum[1/n!*Sum[U[(m^2 + m + 2)/2, 1]*x^(n − 3) +
        Sum[x^(n − t), {t, 4, n}], {m, 1, x − 1}], {n, 0, Infinity}]

        Sum[1/(n!)*
        Sum[1/2*(U[(m^2+m+1)/2, 1] + U[(m^2+m+3)/2, 1])*x^(n − 3) +
        Sum[x^(n − t), {t, 4, n}], {m, 1, x − 1}], {n, 0, Infinity}]
```

Expression (5.10), Binomial identity

```
Sum[Sum[Binomial[n,k]*Binomial[k,j]*(−1)^(k−j)*m^j,
         {j, 0, k}],{k, 0, n}]
```

8. Application 2. An extended version of triangle (1.17)

Row 0:	1										
Row 1:	1	1									
Row 2:	1	7	1								
Row 3:	1	13	13	1							
Row 4:	1	19	25	19	1						
Row 5:	1	25	37	37	25	1					
Row 6:	1	31	49	55	49	31	1				
Row 7:	1	37	61	73	73	61	37	1			
Row 8:	1	43	73	91	97	91	73	43	1		
Row 9:	1	49	85	109	121	121	109	85	49	1	
Row 10:	1	55	97	127	145	151	145	127	97	55	1

Figure 11. Extended version of Triangle (1.17) generated from given $n = 3$ over x from 0 to 10, sequence A287326 in OEIS, [?], [?].