Probability Homework 8 Solution

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February 29, 2016

1 Problem 1

First we consider random 4 vertices in n-vertices graph. Once one of edges is colored, then the remain $\binom{4}{2} - 1 = 5$ edges have the probability $Pr(A_i) = 2^{-5}$ to color to the same color. Where A_i denote the event that clique i is monochromatic in $\binom{n}{4}$ cliques. Also we define that if clique i is monochromatic then random variable $A_i = 1$, otherwise $A_i = 0$. So $E(A_i) = 2^{-5}$.
In order to calculate $E(\sum A_i)$ we yields:

$$E(\sum A_i) = \binom{n}{4}2^{-5}$$

Using the Lemma 6.2 we have $Pr(\sum A_i \leq \binom{n}{4}2^{-5}) > 0$ So there exist one 2-coloring that has at most $\binom{n}{4}2^{-5}$ K_4 are monochromatic. Color the edge independently and uniformly. Denote $X = \sum A_i$. Let $p = Pr(X \leq \binom{n}{4}2^{-5}$. Then we have

$$\begin{aligned}\binom{n}{4}2^{-5} &= E[X] \\ &= \sum_{i \leq \binom{n}{4}2^{-5}} iPr(X = i) + \sum_{i \geq \binom{n}{4}2^{-5}+1} iPr(X = i) \\ &\geq p + (1-p)\binom{n}{4}2^{-5} + 1\end{aligned}$$

So we have

$$\frac{1}{p} \leq \binom{n}{4}2^{-5}$$

Thus, the expected number of samples is at most $\binom{n}{4}2^{-5}$. Testing to see if $X \leq \binom{n}{4}2^{-5}$ can be done in $O(n^4)$ time. So the algorithm can be done in polynomial time.

2 Problem 2

Consider a graph in $G_{n,p}$ with $p = c\frac{\ln n}{n}$. Use the second moment method to prove that if $c < 1$ then, for any constant $\epsilon > 0$ and for n sufficiently large, the graph has isolated vertices with probability at least $1 - \epsilon$.

Solution:

We consider the event X_i denotes that the i^{th} vertex is isolated. So

$$X_i = \begin{cases} 1 & if\ v_i\ is\ isolated \\ 0 & otherwise \end{cases} \tag{1}$$

Let

$$X = \sum_{i=1}^{n} (1-p)^{n-1}. \tag{2}$$

so that

$$E[X] = n(1-p)^{n-1} \tag{3}$$

In order to prove that if $c < 1$ then, for any constant $\epsilon > 0$ and for n sufficiently large, the graph has no isolated vertex with probability at most ϵ. That means $Pr(X = 0) = o(1)$.
We wish to compute

$$Var[x] = Var[\sum_{i=1}^{n} X_i]. \tag{4}$$

Applying Lemma 6.9, we see that we need to consider the covariance of the X_i.

$$\begin{aligned} Cov[X_i X_j] &= E[X_i X_j] - E[X_i]E[X_j] \\ &= (1-p)^{2n-3} - (1-p)^{n-1} * (1-p)^{n-1} \\ &= p(1-p)^{2n-3} \end{aligned} \tag{5}$$

So

$$Var[X] \le E[X] + \sum Cov[X_i X_j] = E[X] + o(pn^2(1-p)^{2n-3}) \tag{6}$$

Then

$$\begin{aligned} Pr(X = 0) &\le \frac{Var[X]}{E[X]^2} \\ &= \frac{1}{n(1-p)^{n-1}} + \frac{p}{1-p} \end{aligned} \tag{7}$$

for $p = c\frac{\ln n}{n}$ and $c < 1$ with $n \to \infty$, $Pr(X = 0) \to o(1)$. So the graph has isolated vertices with probability at least $1 - \epsilon$.

3 Problem 3

Prove the Asymmetric Lovasz Local Lemma: Let $\mathbb{A} = \{A_1, \ldots, A_n\}$ be a set of finite events over a probability space, and for each $1 \le i \le n$, $\tau(A_i) \in \mathbb{A}$ is such that A_i is mutually independent of all events not in $\tau(A_i)$. If $\sum_{A_j \in \tau(A_i)} Pr(A_j) \le 1/4$ for all i, then $Pr(\bigwedge_{i=1}^{n} \bar{A}_i) \ge \prod_{i=1}^{n}(1 - 2Pr(A_i)) > 0$. [Hint: let $x(A_i) = 2Pr(A_i)$ and use the general Lovasz Local Lemma.]

Solution:

First we need to prove a lemma that if $0 \le a_i \le 1/2$ for all $i = 1, 2, \ldots, n$, then $\prod_{i=1}^{n}(1 - 2a_i) \ge 1 - 2\sum_{i=1}^{n} a_i$.
Induction for n. When $n = 1$, the inequality holds obviously. Assume that when $n = k$, the inequality holds. Consider the case when $n = k + 1$,

$$\begin{aligned} \prod_{i=1}^{k+1}(1 - 2a_i) &= \prod_{i=1}^{k}(1 - 2a_i)(1 - 2a_{k+1}) \\ &\ge (1 - 2\sum_{i=1}^{k} a_i)(1 - 2a_{k+1}) \\ &= 1 - 2\sum_{i=1}^{k+1} a_i + 4\sum_{i=1}^{k} a_i a_{k+1} \\ &\ge 1 - 2\sum_{i=1}^{k+1} a_i \end{aligned} \tag{8}$$

So the inequality holds.

Using the general Lovasz Local Lemma, we set $x(A_i) = 2Pr(A_i)$. Then

$$\begin{aligned} x(A_i) \prod_{A_j \in \Gamma(A_i)} (1 - x(A_j)) &= 2Pr(A_i) \prod_{A_j \in \Gamma(A_i)} (1 - 2Pr(A_j)) \\ &\ge 2Pr(A_i)(1 - 2\sum_{A_j \in \Gamma(A_i)} Pr(A_j) \\ &\ge 2Pr(A_i)(1 - 2 * 1/4) \\ &= Pr(A_i) \end{aligned} \tag{9}$$

So the general Lovasz Local Lemma condition holds. Then we have the result

$$
\begin{aligned}
Pr(\bigwedge_{i=1}^{n} \bar{A}_i) &\geq \prod_{i=1}^{n}(1 - x(A_i)) \\
&\geq \prod_{i=1}^{n}(1 - 2Pr(A_i)) \\
&> 0.
\end{aligned}
\tag{10}
$$

4 Problem 4

Given $\beta > 0$, a vertex-coloring of a graph G is said to be β-frugal if (i) each pair of adjacent vertices has different colors, and (ii) no vertex has β neighbors that have the same color.
Prove that if G has maximum degree $\Delta \geq \beta^{\beta}$ with $\beta \geq 2$, then G has a β-frugal coloring with $16\Delta^{1+1/\beta}$ colors. [Hint: you may want to define two types of events corresponding to the two conditions of being β-frugal. Then the result in question 1 can be used.]

Solution:

By the following equation

$$
\binom{\Delta+1}{\beta} = \binom{\Delta}{\beta} + \binom{\Delta}{\beta-1} \tag{11}
$$

we can prove that $\binom{\Delta}{\beta}$ is monotonically increasing for Δ when β is given.
Let the number of colors used to β-frugal coloring be $N = 16\Delta^{1+1/\beta}$, and the algorithm assigns each vertex a uniformly random color.
Now we define two types of events with total number of $m + n$, when n is the number of vertices, and m is the number of edges:

- The pair vertices of e_i has the same color;
- The vertex v_i has β neighbors that have the same color.

Define d_i is the degree of vertex i.
For each event A_i in type I,

$$
Pr(A_i) = \frac{1}{N} \tag{12}
$$

For each event A_i in type II,

$$
\begin{aligned}
Pr(A_i) &= \binom{d_i}{\beta}(\frac{1}{N})^{\beta-1} \\
&\leq \binom{\Delta}{\beta}(\frac{1}{N})^{\beta-1}
\end{aligned}
\tag{13}
$$

Consider the number of dependent events of each event in type I. First, each edge connected to the two vertices in the given edge has an event in type I, whose total number is at most $2(\Delta?1)$. Second, each vertex of the edge has an event in type II, whose total number is exactly 2. Thus, for each event A_i in type I,

$$
\begin{aligned}
\sum_{A_j \in \Gamma(A_i)} Pr(A_j) &\leq 2(\Delta - 1)\frac{1}{N} + 2\binom{\Delta}{\beta}(\frac{1}{N})^{\beta-1} \\
&= 2[(\Delta - 1)\frac{1}{16\Delta^{1+1/\beta}}] + \binom{\Delta}{\beta}(\frac{1}{16\Delta^{1+1/\beta}})^{\beta-1} \\
&\leq 2[\frac{1}{16} + \Delta\dot{(}\Delta - 1) \cdots (\Delta - \beta + 1)]
\end{aligned}
\tag{14}
$$

This definition for events is hard to prove. Another proof from Alistair Sinclair is in the last section.

5 Problem 5