CONTENTS

Chapter 3

Non homogeneous BVP

3.1 Heat Equation

Suppose that r is positive constant. Solve

$$\begin{aligned} ku_{xx} + r = u_t,\ 0 < x < 1,\ t > 0 \\ s.t \\ u(0,t) = 0,\ u(L,t) = u_0,\ t > 0 \\ u(x,o) = f(x),\ 0 < x < 1 \end{aligned}$$

General Solution

$$u(x,t) = -\frac{r}{2k}x^2 + \left(\frac{r}{2k} + u_0\right)x + \sum_{n=1\infty} A_n\, e^{-kn^2\pi^2 t} \sin\, n\pi x$$

where

$$A_n = 2\int_0^1 \left[f(x) + \frac{r}{2k}x^2 - \left(\frac{r}{2k} + u_0\right)\right] \sin\, n\pi x\, dx \tag{3.1}$$

3.2 Exercise 12.6

Problem Solving 3.1. Q1 Use Method1

Solve the equation

$$ku_{tt} = u_t,\ 0 < x < 1,\ t > 0 \tag{3.2}$$

subject to

$$u(0,t) = 100, \;\; u(1,t) = 100$$
$$u(x,0) = 0$$

Solution to Problem Solving 3.1.
By changing the dependent variable u to a new dependent variable ν by substitution

$$u(x,t) = \nu(x,t) + \psi(x) \tag{3.3}$$

Here $r = 0$, $f(x) = 0$. Substitute (3.3) into (3.2) gives

$$k\nu_{xx} + k\psi_{xx} = \nu_t \tag{3.4}$$

Problem A:

$$k\psi^{''} = 0 \tag{3.5}$$
$$\psi^{'} = c_1$$
$$\psi = c_1 x + c_2 \tag{3.6}$$

Apply BC:
$u(0,t) = 100 \rightarrow \psi(0) = 100 \leftarrow$ Since $\nu(0,t) = 0$,

Eqn (3.6) becomes
$100 = c_1(0) + c_2 \rightarrow c_2 = 100$. Thus

$$\psi = c_1 x + 100 \tag{3.7}$$

Now apply BC: $u(1,t) = 100 \rightarrow \psi(1) = 100 \leftarrow$ Since $\nu(1,t) = 0$. Thus eqn (3.7) becomes
$100 = \psi(1) = c_1(1) = 100 \rightarrow c_1 = 0$. Hence

$$\psi(x) = 100 \tag{3.8}$$

Ic: $\nu(x,0) = f(x) - \psi(x)$. But $f(x) = 0$. Now find A_n by using eqn (3.1).

$$\begin{aligned} A_n &= 2\int_0^1 (f(x) - 100)\,\sin n\pi x\,dx \\ &= 2\left[\frac{100}{n\pi}\cos n\pi x\right]_0^1 \\ &= \frac{200}{n\pi}[(-1)^n - 1] \end{aligned} \tag{3.9}$$

Hence the general solution

$$\begin{aligned} u(x,t) &= \psi(x,t) + \nu(x,t) \\ &= 100 + \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{[(-1)^n - 1]}{n}e^{-kn^2\pi^2 t}\sin n\pi x \end{aligned}$$

3.3 Wave Equation

$$\begin{aligned} &a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2},\ 0 < x < L,\ t > 0 \\ &u(0,t) = 0,\ u(L,t) = 0,\ t > 0 \\ &u(x,0) = f(x),\ \frac{\partial u}{\partial t}\Big|_{t=0} = g(x),\ 0 < x < L \end{aligned}$$

Solution

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty}\left(A_n \sin\frac{n\pi at}{L} + B_n \sin\frac{n\pi at}{L}\right)\sin\frac{n\pi x}{L} \\ A_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \\ B_n &= \frac{2}{n\pi a}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx \end{aligned}$$

Problem Solving 3.2. Q9 Exercise 12.6

When a vibrating string is subjected to an external vertical force that varies with the horizontal distance from left to end, the wave equation takes on the form

$$a^2\frac{\partial^2 u}{\partial x^2} + Ax = \frac{\partial^2 u}{\partial t^2} \tag{3.10}$$

where A is a constant. Solve the partial differential equation subject to

$$u(0,t) = 0,\ u(1,t) = 0,\ t > 0 \tag{3.11}$$

$$u(x,0) = 0,\ \frac{\partial u}{\partial t}\Big|_{t=0} = 0,\ 0 < x < 1 \tag{3.12}$$

Solution to Problem Solving 3.2.

Use $u(x,t) = v(x,t) + \psi(x)$ to change (6.24) to new dependent variable v. Eqn (6.24) becomes

$$a^2 \frac{\partial^2 v}{\partial x^2} + a^2 \psi'' + Ax = \frac{\partial^2 v}{\partial t^2}. \tag{3.13}$$

It leads to:
Problem A:

$$a^2 \psi'' + Ax = 0,\ \psi(0) = 0,\ \ \psi(1) = 0 \tag{3.14}$$

Problem B:

$$a^2 \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial t^2} \tag{3.15}$$

$$s.t$$

$$v(0,t) = 0,\ \ v(1,t) = 0 \tag{3.16}$$

$$v(x,0) = -\psi(x), \frac{\partial v}{\partial t}\Big|_{t=0} = 0 \tag{3.17}$$

Solve problem A:

$$\psi'' = -\frac{A}{a^2} x$$

$$\psi' = -\frac{A}{2a^2} x^2 + c_1$$

$$\psi(x) = \frac{A}{6a^2} x^3 + c_1 x + c_2 \tag{3.18}$$

Apply BC $\psi(0) = 0$, eqn (6.28) gives

$$0 = \psi(0) = -\frac{A}{6a^2}(0) + c_1(0) + c_2 \to c_2 = 0$$

$$\psi(x) = -\frac{A}{6a^2} x^3 + c_1 x$$

Apply BC $\psi(1) = 0$, the latest eqn becomes

$$0 = \psi(1) = -\frac{A}{6a^2}(1) + c_1(1) \to c_1 = \frac{A}{6a^2}$$

So,

$$\psi(x) = -\frac{A}{6a^2} x^3 + \frac{A}{6a^2} x \tag{3.19}$$

Solve Problem B by using separation of variables $v(x,t) = X(x)T(t)$. For $\lambda = \alpha^2$, eqn (6.27) that satisfies BC (3.16) and (3.17) gives the solution

$$v(x,t) = \sum_{n=1}^{\infty} (A_n \cos n\pi a t + B_n \sin n\pi t) \sin n\pi x \tag{3.20}$$

Where Here $f(x) = 0$,

$$\begin{aligned}
A_n &= 2\int_0^1 (f(x) - \psi(x)) \sin\, n\pi x\, dx \\
&= \frac{2A}{6a^2}\int_0^1 (x^3 - x) \sin\, n\pi x\, dx \\
&= \frac{2A}{6a^2}\left[\int_0^1 x^3 \sin\, n\pi x\, dx - \int_0^1 x \sin\, n\pi x\, dx\right] \qquad (3.21)
\end{aligned}$$

. Consider:

$$\begin{aligned}
\int x^3 \sin\, n\pi\, x\, dx &= x^3 \int \sin\, n\pi x\, dx - \int [\int \sin\, n\pi x, dx] \frac{d}{dx}(x^3) dx \\
&= \frac{x^3}{n\pi}(-\cos\, n\pi x) + \frac{1}{n\pi}\int 3x^2 \cos\, n\pi x\, dx \\
&= -\frac{x^3}{n\pi}\cos\, n\pi x + \frac{3}{n\pi}\left[\frac{x^2}{n\pi}\sin\, n\pi x - \int \frac{1}{n\pi}(\sin\, n\pi x)\frac{d}{dx}(x^2) dx\right] \\
&= -\frac{x^3}{n\pi}\cos\, n\pi x + \frac{3x^2}{n^2\pi^2}\sin\, n\pi x - \frac{6}{n^2\pi^2}\left[\int x \sin\, n\pi x\, dx\right] \\
&= -\frac{x^3}{n\pi}\cos\, n\pi x + \frac{3x^2}{n^2\pi^2}\sin\, n\pi x \\
&- \frac{6}{n^2\pi^2}\left[x\int \sin\, n\pi x\, dx - \int [\int \sin\, n\pi x\, dx]\right] \\
&= \frac{x^3}{n\pi}\cos\, n\pi x + \frac{3x^2}{n^2\pi^2}\sin\, n\pi x \\
&- \frac{6}{n^2\pi^2}\left[-\frac{x}{n\pi}\cos\, n\pi x + \frac{1}{n\pi}\int \cos\, n\pi x\, dx\right] \\
&= \frac{x^3}{n\pi}\cos\, n\pi x + \frac{3x^2}{n^2\pi^2}\sin\, n\pi x \\
&+ \frac{6x}{n^3\pi^3}\cos\, n\pi x - \frac{6}{n^4\pi^4}\sin\, n\pi x \\
\int_0^1 x^3 \sin\, n\pi x\, dx &= \left[\frac{x^3}{n\pi}\cos\, n\pi x + \frac{3x^2}{n^2\pi^2}\sin\, n\pi x + \frac{6x}{n^3\pi^3}\cos\, n\pi x - \frac{6}{n^4\pi^4}\sin\, n\pi x\right]_0^1 \\
&= -\frac{1}{n\pi}(-1)^n + \frac{6}{n^3\pi^3}(-1)^n \\
&= -\frac{1}{n\pi}(-1)^n + \frac{2}{n^3\pi^3}((-)^n - 1) \\
\int_0^1 x \sin\, n\pi x\, dx &= [x(-\frac{1}{n\pi}\cos\, n\pi x) + \frac{1}{n^2\pi^2}\sin\, n\pi x]_0^1 \\
&= -\frac{1}{n\pi}(-1)^n
\end{aligned}$$

Substitute into (3.42) gives

$$A_n = \frac{2A(-1)^n}{n^3\pi^3 a^2} \tag{3.22}$$

Since $g(x) = 0$, thus $B_n = 0$. From (3.43),

$$v(x,t) = \frac{2A}{a^2\pi^3}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^3}\right]\cos n\pi at \sin n\pi x \tag{3.23}$$

Solution:

$$\begin{aligned} u(x,t) &= \psi(x) + v(x,t) \\ &= \frac{A}{6a^2}(x - x^3) + \frac{2A}{a^2\pi^3}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^3}\right]\cos n\pi at \sin n\pi x \end{aligned}$$

Q2 Dec 2014

Consider the following BVP

$$u_{tt}(x,t) = \frac{1}{25}u_{xx}(x,t) + \sin\frac{x}{2},\ \ 0 < x < 1,\ \ t > 0 \tag{3.24}$$

$$u(0,t) = 0,\ t > 0 \tag{3.25}$$

$$u_x(1,t) = 0,\ t > 0 \tag{3.26}$$

$$u_t(x,0) = 0,\ 0 < x < \pi \tag{3.27}$$

$$u(x,0) = 200\sin\frac{x}{2},\ 0 < x < 1 \tag{3.28}$$

a) Interpret the boundary and initial conditions.
b) Determine $u(x,t)$ for $t > 0$.

Solution ;;

a) IC (3.28)($f(x) = 200\sin\frac{x}{2}$) denotes the initial vertical displacement (transverse vibration) distribution throughout.
IC (3.27) denotes the initial velocity is zero (release from rest).
BC: $u(0,t) = 0$ means that displacement zero at $x = 0$.
BC: $u_x(1,t) = 0$ is called **free-end-condition**
b) Let

$$u(x,t) = v(x,y) + \psi(x) \tag{3.29}$$

Substitute (3.29)into pde gives

$$\begin{aligned} u_{tt} &= \frac{1}{25}u_{xx} + \sin\frac{x}{2} \\ \frac{\partial^2 v}{\partial t^2} &= \frac{1}{25}\left(\frac{\partial^2 v}{\partial x^2} + \psi''\right) + \sin\frac{x}{2} \end{aligned}$$

gives ODE and homogeneous PdE.

$$\frac{1}{25}\psi'' + \sin\frac{x}{2} = 0 \tag{3.30}$$

$$\frac{\partial^2 v}{\partial t^2} = \frac{1}{25}\frac{\partial^2 v}{\partial x^2} \tag{3.31}$$

BC:$u(0,t) = v(0,t) + \psi(0) = 0$ gives $\psi(0) = 0 \leftarrow$ Since $v(0,t) = 0$

BC:$=u_x(1,t) = v_x(1,t) + \psi_x(1) = 0$ gives $\psi_x(1) = 0 \leftarrow$ Since $v_x(1,t) = 0$

Solve eqn (3.30),

$$\psi'' = -25 \sin\frac{x}{2}$$

$$\psi' = -25\frac{1}{1/2}(-\cos\frac{x}{2}) + c_1$$

$$= 50 \cos\frac{x}{2} + c_1$$

$$\psi(x) = 100 \sin\frac{x}{2} + c_1 x + c_2$$

Apply BC:$\psi(0) = 0$ gives
$0 = \psi(0) = 100 \sin\frac{0}{2} + c_1(0) + c_2 \to c_2 = 0$. Thus

$$\psi(x) = 100 \sin\frac{x}{2} + c_1 x \tag{3.32}$$

$$\psi_x(x) = 50 \cos\frac{x}{2} + c_1 \tag{3.33}$$

Next apply BC: $\psi_x(1) = 0$
$\psi_x(1) = 0 = 50 \cos\frac{1}{2} + c_1(1) \to c_1 = -50 \cos\frac{1}{2} = -43.9$.
The solution is

$$\psi(x) = 100 \sin\frac{x}{2} - 43.9x \tag{3.34}$$

For PDE (3.31) subject to
$v(0,t) = 0,\ v_x(1,t) = 0,\ 0 < x < 1$
$v(x,0) = 200 \sin\frac{x}{2} - \psi(x)$
Solve (3.31) by using the method of separation of variables. For cases $\lambda = 0$ and $\lambda = -\alpha^2$ give the trivial solution. Now for $\lambda = \alpha^2$, gives
$v(x,t) = \sum_{n=1}^{\infty} (A_n \cos n\pi at + B_n \sin n\pi at) \sin n\pi x$
Apply IC: $v_t(x,t) = 0, t = 0$

$$\frac{\partial v}{\partial t} = \sum_{n=1}^{\infty} (-n\pi a A_n \sin n\pi at + n\pi B_n \cos n\pi at) \sin n\pi x$$

$$0 = v_t(x,0) = \sum_{n=1}^{\infty} (-n\pi a B_n) \sin n\pi x$$

Half-range of 0 of sine series will give $B_0 = 0$. Thus

$$v(x,t) = \sum_{i=1}^{\infty} (A_n \cos n\pi at) \sin n\pi x \tag{3.35}$$

Now apply IC: $u(x,0) = 200 \sin \frac{x}{x} \to v(x,0) = 200 \sin \frac{x}{2} - \psi$,

$$v(x,0) = 200 \sin \frac{x}{2} - (100 \sin \frac{x}{2} - 43.9) = \sum_{n=1}^{\infty} A[n] \sin n\pi x$$

Half-range of $100 \sin \frac{x}{2} + 43.9$ of sine series will give A_n,

$$\begin{aligned}
A_n &= \frac{2}{\pi}\int_0^1 (43.9 + 100 \sin \frac{x}{2}) \sin n\pi\, x\, dx \\
&= \frac{2}{\pi}\left[\left(\frac{43.9}{n\pi}\right)(-\cos n\pi x)\right]_0^1 + \frac{200}{\pi}\int_0^1 \frac{1}{2}[\cos(\frac{1}{2} - n\pi)x + \cos(\frac{1}{2} + n\pi)x]dx \\
&= \frac{2}{\pi}\left[\left(\frac{43.9}{n\pi}\right)(-\cos n\pi x)\right]_0^1 + \frac{100}{\pi}\left[\frac{1}{(1/2 - n\pi)}\sin(1/2 - n\pi)x + \frac{1}{(1/2 + n\pi)}\sin(1/2 + n\pi)x\right]_0^1 \\
&= \frac{87.8}{n\pi^2}(1 - (-1)^n) + \frac{1}{(1/2 - n\pi)}(-\sin\frac{1}{2}) + \frac{1}{(1/2 + n\pi)}(-\sin\frac{1}{2}) \\
&\leftarrow \boxed{\text{use } \cos(A+B) + \cos(A-B) = 2\sin A \sin B} \\
&= \frac{87.8}{n\pi^2}(1 - (-1)^n) - \frac{1}{\frac{1}{4} - n^2\pi^2}\sin\frac{1}{2} \\
&= \frac{87.8}{n\pi^2}(1 - (-1)^n) - \frac{0.5}{1/4 - n^2\pi^2} \\
&= \frac{87.8}{n\pi^2}(1 - (-1)^n) - \frac{2}{1 - n^2\pi^2}
\end{aligned}$$

Hence the solution is given by

$$u(x,t) = \psi(x) + v(x,t) \tag{3.36}$$

$$= 100 \sin \frac{x}{2} - 43.9 + \sum_{n=1}^{\infty}\left(\frac{87.8}{n\pi^2}(1 - (-1)^n) - \frac{2}{1 - n^2\pi^2}\right)\cos n\pi at \sin n\pi x \tag{3.37}$$

3.4 Laplace Equation

Standard Formula

$$u_{xx} + u_{yy} = 0,\ 0 < x < a,\ 0 < y < b \tag{3.38}$$
$$s.t \tag{3.39}$$

,

$$\frac{\partial u}{\partial x}\Big|_{x=0} = 0,\ \frac{\partial u}{\partial x}\Big|_{x=a} = 0, , 0 < y < b \tag{3.40}$$
$$u(x,0) = 0,\ u(x,b) = f(x),\ 0 < x < a \tag{3.41}$$

Solution :

$$u(x,t) = A_0 y + \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi}{a} y \cos \frac{n\pi}{a} x \tag{3.42}$$
$$A_n = \frac{2}{a \sinh \frac{n\pi}{a} b} \int_0^a f(x) \cos \frac{n\pi}{a} x \, dx \tag{3.43}$$

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$$u_{xx} + u_{yy} = 0,\ 0 < x < 1,\ 0 < y < 1 \tag{3.44}$$
$$s.t \tag{3.45}$$

,

$$\frac{\partial u}{\partial y}\Big|_{y=0} = 0,\ \frac{\partial u}{\partial y}\Big|_{y=1} = 0, , 0 < x < 1 \tag{3.46}$$
$$u(0,x) = 0,\ u(1,y) = f(y),\ 0 < y < 1 \tag{3.47}$$

Solution : Use the separation of variable method, $u(x,y) = X(x)Y(y)$. Substitute into pde (3.44),

$$X^{''}Y + XY^{''} = 0$$
$$\frac{Y^{''}}{Y} = -\frac{X^{''}}{X} = -\lambda$$

leads to two ODEs.

$$Y^{''} + \lambda Y = 0 \tag{3.48}$$
$$X^{''} - \lambda X = 0 \tag{3.49}$$

For $\lambda = 0$ gives $Y(y) = c_1$ and $\lambda = -\alpha^2$ gives trivial solution. Now for $\lambda = \alpha^2$, and translate BC into $Y^{'} = 0$ and $Y^{'}(1) = 0$, (3.48) becomes

$$Y^{''} + \alpha^2 Y = 0,\ Y^{'}(0) = 0,\ Y^{'}(1) = 0 \tag{3.50}$$

Solve (3.50) gives

$$Y = c_1 \cos \alpha y + c_2 \sin \alpha y \tag{3.51}$$

Apply BC: $Y^{'}(0) = 0$,

$$\begin{aligned} y' &= -c_1\alpha \sin \alpha y + c_2\alpha \cos \alpha y \\ 0 &= c_1\alpha(1) \rightarrow c_2 = 0. \end{aligned}$$

So (3.51) becomes

$$Y = c_1 \cos \alpha y \tag{3.52}$$

$$Y^{'} = -c_1\alpha \sin \alpha y \tag{3.53}$$

Use BC: $Y^{'}(1) = 1$. For non trivial $C_1 \neq 0$, $\sin \alpha y = 0 \rightarrow \alpha = n\pi$. Thus for $n = 0$, and $n \geq 1$, the eigenfunction of (3.50) are

$Y = c_1$, $n = 0$ and $Y = c_1 \cos n\pi y,\ n = 1, 2, \ldots,$.
Now by interchange $x \leftrightarrow y$, use (3.42),the solution is

$$u(x, y) = A_0 x + \sum_{n=1}^{\infty} A_n \sinh n\pi x \cos n\pi y \tag{3.54}$$

Where, from (3.43)

$$A_n = \frac{2}{\sinh n\pi} \int_0^1 f(y) \cos n\pi y \, dy$$

Chapter 4

Orthogonal Series Expansion

- For a certain types of boundary condition s the method of separation of variables and the superposition principle to to an expansion in a trigonometric series that is not a Fourier series.
- To solve the problem in this section , we utilize the concept of orthogonal series expansion or generalized Fourier series.

4.1 Using Orthogonal series expansion

VBP :

The temperature in a rod of a unit length in which heat transfer from its right boundary into a surrounding medium at a constant temperature zero is determine from

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},\ 0 < x < 1,\ \ t > 0$$
$$u(0,t) = 0,\ \frac{\partial u}{\partial x}\Big|_{x=1} = -hu(1,t),\ h > 0,\ t > 0$$
$$u(x,0) = 1,\ 0 < x < 1$$

Solution

Using separation of variables with $u(x,t) = X(x)T(t)$, we find

$$kX^{''}T = XT^{'}$$
$$\frac{X^{''}}{X} = \frac{1}{k}\frac{T^{'}}{T} = -\lambda$$

leads to the separated equation with BC respectively,

$$X^{''} + \lambda X = 0 \tag{4.1}$$
$$T^{'} + \lambda T = 0 \tag{4.2}$$

with BC: $X(0) = 0$ and $X^{'}(1) + hX(1) = 0$.
Solve (4.1); for $\lambda = 0, -\alpha^2 < 0$ will yield trivial solution.
For $\lambda = \alpha^2$, eqn (4.1) will yield

$$x(x) = c_1 \cos \alpha x + c_2 \sin \alpha x \tag{4.3}$$

Apply BC: $X(0) = 0$,

$$X(0) = c_1 \cos(0) + c_2 \sin(0) = 0$$
$$c_1 = 0$$
$$X(x) = c_2 \sin \alpha x \tag{4.4}$$

Apply BC: $X^{'}(1) + hX(1) = 0$ on (4.4),

$$X^{'}(1) = \alpha c_2 \cos \alpha(1)$$
$$hX(1) = h(c_2 \sin \alpha$$
$$X^{'}(1) - hX(1) = c_2(\alpha \cos \alpha + h \sin \alpha = 0$$
$$\alpha \cos \alpha + h \sin \alpha = 0$$
$$\tan \alpha = -\frac{\alpha}{h} \leftarrow$$ has an infinite number of roots-see section 11.4

If the consecutive positive roots are denoted $a_n, n = 1, 2, \ldots$ then the eigenvalues of the problem are $\lambda_n = \alpha_n^2$ corresponding to eigenfunctions are

$$X(x) = c_2 \sin \alpha_n x \, n = 1, 2, 3 \ldots$$

The solution of DE (4.2) is

$$T(t) = c_3 e^{-k\alpha_n^2 t}$$

and so

$$u_n = X(x)T(t) = A_n \, e^{-k\alpha_n^2 t} \sin \alpha_n x \tag{4.5}$$
$$and \tag{4.6}$$
$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\alpha_n^2 t} \sin \alpha x \tag{4.7}$$

Now apply IC;at t=0, $u(x, 0) = 1, 0 < x < 1$, so that

$$1 = \sum_{n=1}^{\infty} A_n \sin \alpha_n x \tag{4.8}$$

The series (4.8) is not Fourier sine series. It is an expansion of $u(x, 0) = 1$ in term of orthogonal functions. It follows that the set of eigenfunctions $\{\sin \alpha_n x\}$, $n = 1, 2, 3 \ldots$

where $a's$ are defineed by tan $\alpha = -\frac{\alpha}{h}$ is orthogonal with respect to the weight function $p(x) = 1$. So

$$A_n = \frac{\int_0^1 \sin\,\alpha_n x\,dx}{\int_0^1 \sin^2\,\alpha_n^2 x\,dx} \tag{4.9}$$

To evaluate:

$$\begin{aligned}\int_0^1 \sin_n^2 x\,dx &= \frac{1}{2}\int_0^1 (1-\cos\,2\alpha x)dx \\ &= \frac{1}{2}\left(1-\frac{1}{2\alpha_n}\sin\,2\alpha_n\right) \qquad (4.10)\end{aligned}$$

Using

$$\sin\,2\alpha_n = 2\,\sin\,\alpha_n\,\cos\,\alpha_n \tag{4.11}$$

$$a_n\,\cos_,\alpha_n = -h\,\sin\,\alpha_n \tag{4.12}$$

(4.10) becomes

$$\int_0^1 \sin^2\alpha_n\,dx = \frac{1}{2}\left(1-\frac{1}{2\alpha_n}\sin\,2\alpha_n\right) \tag{4.13}$$

$$= \frac{1}{2}(1-\frac{1}{2\alpha_n}2\,\sin\,\alpha_n\,\cos\,\alpha_n) \tag{4.14}$$

$$= \frac{1}{2}(1-(h\,\cos^2\,\alpha_n) \tag{4.15}$$

$$= \frac{1}{2h}(h+\cos^2\alpha_n) \tag{4.16}$$

$$\int_0^1 \sin\,\alpha_n x\,dx = \frac{1}{\alpha_n}(1-\cos\,\alpha_n) \tag{4.17}$$

Consequently (4.9) becomes

$$A_n = \frac{2h(1-\cos\,\alpha_n)}{\alpha_n(h+\cos^2\,\alpha_n)}$$

Finally the solution of BVP is

$$u(x,t) = 2h\sum_{n=1}^{\infty}\frac{1-\cos\,\alpha_n}{\alpha_n(h+\cos^2\alpha_n)}e^{-k\alpha_n^2 t}\sin\,\alpha_n x \tag{4.18}$$

4.1.1 Summary

BVP

$$u_t(x,t) = a^2 u_{xx}(x,t),\ 0 < x < L,\ \ t > 0 \tag{4.19}$$
$$u(0,t) = 0,\ t > 0 \tag{4.20}$$
$$u_x(L,t) + hU(L,t) = 0,\ t > 0 \tag{4.21}$$
$$u(x,o) = f(x),\ 0 < x < L \tag{4.22}$$

General solution

$$u(x,t) = \sum_{n=1}^{\infty} A_n\, e^{-\left(\frac{z_n \alpha}{L} t\right)^2} \sin \frac{z_n x}{L} \leftarrow \boxed{z_n = \alpha_n}$$

Q1 Exercises 12.7 In example 1 find the temperature $u(x,t)$ when the left end of the rod is insulated.

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},\ \ 0 < x < 1,\ t > 0 \tag{4.23}$$
$$\frac{\partial u}{\partial x}\Big|_{x=0} = 0,\ \frac{\partial u}{\partial x}\Big|_{x=1} = -hu(1,t),\ h > 0,\ t > 0 \tag{4.24}$$
$$u(x,0) = 1,\ \ 0 < x < 1 \tag{4.25}$$

Solution :

Let $u(x,t) = X(x)T(t)$. The substitute into (4.23) gives

$$X^{''} + \lambda X = 0 \tag{4.26}$$
$$T^{''} + k\lambda T = 0 \tag{4.27}$$
$$X^{'}(0) = 0 \text{ and } X(0) = 0,\ X^{'}(1) + h(X(1) = 0 \tag{4.28}$$

Solve (4.26):
For $\lambda = 0$ and $\lambda = -\alpha^2 < 0$ give $u(x,t) = 0$ (trivial solution). For $\lambda = \alpha^2$,

$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x \tag{4.29}$$

Apply BC $X^{'}(0) = 0$ gives

$$X^{'}(0) = -c_1 \alpha \sin \alpha(1) + c_2 \alpha \cos \alpha(1) = 0 \rightarrow C_2 = 0$$

So

$$X(x) = c_1 \cos \alpha\, x \tag{4.30}$$

Apply the second BC of (4.28) to (4.30) yields

$$\begin{aligned}
X'(x) &= -c_1\alpha \sin \alpha x \\
X'(1) &= -c_1\alpha \sin \alpha \\
hX(1) &= hc_1 \cos \alpha \\
X'(1) + hX(1) &= -c_1\alpha \sin \alpha + c_1 h \cos \alpha = 0 \\
&= c_1(-\alpha \sin \alpha + h \cos \alpha) = 0 \\
-\alpha \sin \alpha + h \cos \alpha = 0 & \\
\text{or} & \\
\tan \alpha &= \frac{h}{\alpha}
\end{aligned}$$

The last eqn has an infinite number of roots. If the positive roots are denoted by α_n, $n = 1, 2, 3 \ldots$, the the eigenvalues of the problem are $\lambda_n = \alpha_n^2$. The corresponding eigenfunctions are

$$X(x) = c_1 \cos \alpha_n x, \;\; n = 1, 2, 3 \ldots \tag{4.31}$$

Solve ODE (4.27) gives $T = e^{-k\alpha_n^2 t}$, and so

$$\begin{aligned}
u_n &= XT = A_n e^{-k\alpha_n^2 t} \cos \alpha_n x \\
\text{and} & \\
u(x,t) &= \sum_{n=1}^{\infty} A_n e^{-k\alpha_n^2 t} \cos \alpha_n x
\end{aligned} \tag{4.32}$$

Now apply IC $u(x, 0) = 1$. At $t = 0$ so that

$$u(x, 0) = 1 = \sum_{n=1}^{\infty} A_n \cos \alpha_n x \tag{4.33}$$

The series in (4.33) is an expansion of $u(x, 0) = 1$ in terms of orthogonal function. It follows that the set eigenfunctions $\{\cos \alpha_n x\}$, $n = 1, 2, \ldots$ where $\alpha_n' s$ are defined by

$\tan\,\alpha = h/\alpha$ is orthogonal with respect to $p(x) = 1$. It follows that

$$A_n = \frac{\int_0^1 \cos\,\alpha_n x\,dx}{\int_0^1 \cos^2\,\alpha_n x\,dx}$$

$$\int_0^1 \cos\,\alpha_n x\,dx = \frac{1}{\alpha_n}[\sin\,\alpha_n x]_0^1$$
$$= \frac{1}{\alpha_n}\sin\,\alpha_n$$

$$\int_0^1 \cos^2\,\alpha x\,dx = \frac{1}{2}\int_0^2 (1+\cos\,2\alpha_n x)dx$$
$$= \frac{1}{2}\left[x + \frac{1}{2\alpha_n}\sin\,2\alpha_n x\right]_0^1 \leftarrow \boxed{\cos\,x^2 = \tfrac{1+\cos\,2x}{2}}$$
$$= \frac{1}{2}\left[1 + \frac{1}{2\alpha_n}\sin\,2\alpha_n\right]$$
$$= \frac{1}{2}\left[1 + \frac{1}{2\alpha_n}2\,\sin\,\alpha\,\cos\,\alpha\right] \leftarrow \boxed{\text{use } \cos\,2x = 2\,\sin\,x\,\cos\,x}$$
$$= \frac{1}{2}\left[1 + \frac{1}{\alpha}\sin\,\alpha_n(\frac{\alpha}{h}\sin\,\alpha_n)\right] \leftarrow \boxed{\cos\,\alpha = \tfrac{\alpha}{h}\sin\,\alpha}$$
$$= \frac{1}{2h}(h+\sin^2\alpha)$$
$$= \frac{2h}{\alpha_n}\frac{\sin\,\alpha_n}{h+\sin^2\,\alpha_n}$$

Hence the general solution is

$$u(x,t) = 2h\sum_{n=1}^{\infty}\frac{\sin\,\alpha_n}{\alpha_n(h+\sin^2\,\alpha_n)}e^{-k\alpha_n^2 t}\cos\,\alpha_n x$$

4.2 Past Sem Paper

Problem Solving 4.1. Q3 JUN 2012

Consider the following boundary-value problem:

$$u_t(x,t) = u_x x(x,t) + 2,\ 0 < x < 1,\ t > 0$$
$$u(0,t) = 0,\ u_x(1,t) + u(1,t) = 0,\ t > 0$$
$$u(x,t) = x+1,\ 0 < x < 1$$

Chapter 5

Higher Dimensional Problems

5.1 Two-Dimensional Linear PDE that represent Temperature

Given

$$k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \frac{\partial u}{\partial t},\ 0 < x < b,\ 0 < y < c,\ t > 0$$

$$s.t$$

$$u(0, y, t) = 0,\ u(b, y, t) = 0,\ 0 < y < c,\ \ t > 0$$
$$u(x, 0, t) = 0,\ u(x, c, t) = 0,\ 0 < y < c,\ t > 0$$
$$u(x, y, 0) = f(x, y),\ 0 < x < b,\ \ 0 < y < c$$

Solution

Apply separation of variables

$$u(x, y, t) = X(x)Y(y)T(t) \tag{5.1}$$

Substitute (**??**) into PDE gives

$$k(X^{''}YT + XY^{''}T) = XYT^{'}$$

or

$$\frac{X^{''}}{X} = -\frac{Y^{''}}{Y} + \frac{T^{'}}{kT} \tag{5.2}$$

Both sides of eqn (5.2) must equal to a constant $-\lambda$:

$$X^{''} + \lambda X = 0 \tag{5.3}$$

$$\frac{Y^{''}}{Y} = \frac{T^{'}}{KT} + \lambda \tag{5.4}$$

Introduce another separation constant $-\mu$ in (5.4) becomes

$$\frac{Y''}{Y} = -\mu \text{ and } \frac{T'}{kT} + \lambda = -\mu$$
$$Y'' + \mu Y = 0 \text{ and } T' + k(\lambda + \mu)T = 0 \tag{5.5}$$

Now homogenoues BC:

$$\left.\begin{array}{ll} u(0,y,t) = 0, & u(b,y,t) = 0 \\ u(x,0,t) = 0, & u(x,c,t) = 0 \end{array}\right\} \rightarrow \left\{\begin{array}{ll} X(0) = 0, & X(b) = 0 \\ Y(0) = 0, & Y(c) = 0 \end{array}\right.$$

Thus we have two Sturm-Liouville problems:

$$X'' + \lambda X = 0,\ X(0) = 0,\ X(b) = 0 \tag{5.6}$$
$$Y'' + \mu Y = 0,\ Y(0) = 0,\ Y(c) = 0 \tag{5.7}$$

Solve (5.6):-
Case 1: $\lambda = 0$, $\mu = 0$:

$$X(x) = c_1 x + c_2 \tag{5.8}$$

Apply BC, $X(0) = 0$ on (5.8) gives

$$X(0) = 0 = c_1(0) + c_1 \rightarrow c_2 = 0$$

So $X(x) = c_1 x$. Apply BC, $X(b) = 0$ gives $0 = c_1(0) \rightarrow c_1 = 0$ Similarly for (5.7) will give $Y(y) = 0$ when $\mu = 0$.
Case 2: $\lambda = -\alpha^2 < 0$, $\mu = -\alpha^2 < 0$:
Eqns (5.6) and (5.7) become
$X(x) = c_3 \cosh \alpha x + c_4 \sinh \alpha x$ and $Y(y) = c_5 \cosh \alpha y + c_6 \sinh \alpha y$ will give the trivial solution when apply BCs.
<u>Case 3:</u>$\lambda = \alpha^2 > 0$, $\mu = \alpha^2 > 0$:
The solution of eqns (5.6) and (5.7) are

$$X(x) = c_7 \cos \alpha x + c_8 \sin \alpha x \tag{5.9}$$
$$Y(y) = c_9 \cos \alpha x + c_{10} \sin \alpha x \tag{5.10}$$

When apply BC on (5.9) gives

$$X(0) = 0 = c_7(1) \rightarrow c_7 = 0$$
$$X(x) = c_8 \sin \alpha x$$

Apply BC, $Y(b) = 0$, give the nontrivial solution: $c_8 \neq 0$, $\sin \alpha(b) = 0 \rightarrow \alpha = \frac{n\pi}{b}$. So the eigenvlue, $\lambda_m = \frac{m^2\pi^2}{b^2}$ and the corresponding eigenfunction

$$X(x) = c_8 \sin \frac{m\pi}{b} x \tag{5.11}$$

Similarly , the eigenvlue and eigenfunction of (5.10) are

$$\lambda_n = \frac{n^2\pi^2}{c^2} \tag{5.12}$$

$$Y(y) = c_{10} \sin \frac{n\pi}{c} \tag{5.13}$$

Now solve (5.5),

$$T' + k(\frac{m^2\pi}{b^2} + \frac{n^2\pi^2}{c^2})T = 0$$

$$T = c_{11} e^{-k(\frac{m^2\pi}{b^2} + \frac{n^2\pi^2}{c^2})t}$$

A product solution of the two-dimensional heat equation,

$$u_{mn}(x, y, t) = A_{mn} e^{-(\frac{m^2\pi}{b^2} + \frac{n^2\pi^2}{c^2})t} \sin \frac{m\pi}{b} x \sin \frac{n\pi}{c} y$$

where A_{mn} is an arbitrary constant.
By the superposition principle, the solution,

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} e^{-k(\frac{m^2\pi}{b^2} + \frac{n^2\pi^2}{c^2})t} \sin \frac{m\pi}{b} x \sin \frac{n\pi}{c} y \tag{5.14}$$

Now apply IC; at $t = 0$, eqn (5.15) gives

$$u(x, y, 0) = f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi}{b} x \sin \frac{n\pi}{c} y$$

$$A_{mn} = \frac{4}{bc} \int_0^c \int_0^b f(x, y) \sin \frac{m\pi}{b} x \sin \frac{n\pi}{c} y \, dx \, dy \tag{5.15}$$

Thus the solution of the BVP consists of (5.14) with A_{mn} given by (5.15)

5.2 BVP subjected to the following BC

$$u_t(x, y, t) = a^2[u_{xx}(x, y, t) + u_{yy}(x, y, t)], \; 0 < x < b, \; 0 < y < c, \; t > 0$$

$s.t$

$$u_x(0, y, t) = u_x(b, y, t) = 0,], 0 < y < c, \; t > 0$$

$$u_y(x, 0, t) = u)y(x, c, t) = 0, \; 0 < x < b, \; t > 0$$

$$u(x, y, 0) = f(x, y), \; 0 < x < b, \; 0 < y < c$$

Solution

$$u(x,y,t) = A_{00} + \sum_{m=1}^{\infty} A_{m0} \cos\left(\frac{m\pi x}{b}\right) + \sum_{n=1}^{\infty} A_{0n} \cos\left(\frac{n\pi y}{c}\right)$$
$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \cos\left(\frac{m\pi x}{b}\right) \cos\left(\frac{n\pi y}{c}\right)$$
$$A_{00} = \frac{1}{bc}\int_0^c$$

5.3 Suggested answers

5.3.1 Q4 Jun 2014

Problem Solving 5.1. Consider the following two-dimensional heat equation

$$u_t(x,y,t) = u_{xx}(x,y,t) + u_{yy}(x,y,t),\ 0 < x < 2\pi,\ 0 < y < 2\pi,\ t > 0$$
$$u(0,y,t) = u(2\pi,y,t) = 0,\ 0 < y < 2\pi$$
$$u(x,0,t) = u(x,2\pi,t) = 0,\ 0 < x < 2\pi$$
$$u(x,y,0) = x\, sin\, y,\ 0 < y < 2\pi$$

a) Interpret the boundary-value problem above.

b) Determine the temperature distribution $u(x,y,t)$.

Solution

a) BVP describe the problem of temperature over the rectangle $0 \le x \le \pi$ by $0 \le y \le \pi$ whose the initial temperature is x sin y throughout and boundaries are held at temperature zero for time $t > 0$.

b) By using the method of the separation of variable $u(x,y,t) = X(x)Y(y)T(t)$ and consider the $\lambda = \alpha^2 > 0$ and $\mu = \alpha^2 > 0$ we obtain the aigenvalues and theirs corresponding eigenfunctions

$$\lambda_m = \frac{m\pi}{2\pi} = \frac{m}{2}\ \ \mu_n = \frac{n\pi}{2\pi} = \frac{n}{2}$$

and the corresponding eigenfunctions

$$X(x) = c_1 \sin \frac{m}{2}x,\ Y(y) = c_2 \sin \frac{n}{2}y$$

respectively, where $m = 1, 2, 3\dots$ and $n = 1, 2, 3, \dots$
A product solution of the 2-D heat equation that satisfies BC

$$u_{mn} = A_{mn} e^{-(\frac{m^2}{4}+\frac{n^2}{4})} \sin \frac{m}{2}x \sin \frac{n}{2}y \tag{5.16}$$

where A_{mn} is an arbitrary constant.
By the superposition principle in the form of a double sum

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} e^{-(\frac{m^2+n^2}{4})} \sin \frac{m}{2}x \sin \frac{n}{2}y \tag{5.17}$$

To find A_{mn}:
At $t = 0$

$$x \sin y = u(x, y, 0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m}{2}x \sin \frac{n}{2}y$$

We can find the coefficient A_{mn} by using the double integration,

$$\begin{aligned}
A_{mn} &= \frac{4}{2\pi 2\pi} \int_0^{2\pi} \int_0^{2\pi} x\, sin\, y \sin \frac{m}{2}x \sin \frac{n}{2}y \, dx\, dy \\
&= \frac{1}{\pi^2} \int_0^{2\pi} \left[-\frac{2x}{m} \cos \frac{m}{2}x + \frac{4}{m^2} \sin \frac{m}{2}x \right]_0^{2\pi} \sin \frac{m}{2}y \, dy \\
&= -\frac{4}{\pi}(-1)^m \int_0^{2\pi} \sin \frac{m}{2}y \, dy \\
&= -\frac{4}{m\pi}(-1)^m \left[-\frac{2}{m} \cos \frac{m}{2}y \right]_0^{2\pi} \\
&= \frac{8}{m^2\pi}(-1)^m((-1)^m - 1)
\end{aligned}$$

The solution is given by (5.17) where $A_{mn} = \frac{8}{m^2\pi}(-1)^m((-1)^m - 1)$,

Chapter 6

VBP In Polar coordinate

6.1 Exercise 13.1/ Q5.Jun 2012

Problem Solving 6.1. Q13.9

Solve VBP

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,\ 0 < \theta < 2\pi,\ a < r < b$$

$$u(a,\theta) = f(\theta),\ u(b,\theta) = 0$$

Solution

a) Defining $u(r,\theta) = R(r)\Theta(\theta)$ and separating variable gives,

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

$$r^2R''\Theta + rR'\Theta + R\Theta'' = 0 \leftarrow \boxed{\text{multiply by } r}$$

$$r^2R''\Theta + rR'\Theta = -R\Theta''$$

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

The separate equations are

$$r^2R'' + rR' - \lambda R = 0 \tag{6.1}$$

$$\Theta'' + \lambda\Theta = 0 \tag{6.2}$$

Solve (6.2) of the problem

$$\Theta'' + \lambda\Theta = 0,\ \ \Theta(\theta) = \Theta(\theta + 2\pi) \tag{6.3}$$

For $\lambda = 0$:

$$\Theta(\theta) = c_1 + c_2\theta \tag{6.4}$$

For $\lambda = -\alpha < 0$.

$$\Theta(\theta) = c_1 \cosh\ \alpha\theta + c_2 \sinh\ \alpha\theta \tag{6.5}$$

For $\lambda = \alpha^2 > 0$,

$$\Theta(\theta) = c_1 \cos\alpha\theta + c_2 \sin\alpha\theta \tag{6.6}$$

Solution (6.4) is nonperiodic unless $C_2 = 0$. Similarly, solution (6.5) is non periodic unless $c_1 = 0$ and $c_2 = 0$. Solution (6.6) will be 2π-periodic if we take $\alpha = n$. the eigenvalues of (6.1) are then $\lambda_0 = 0$ and $\lambda_n = n^2$, $n = 1, 2, 3.\ldots$. If we correspond $\lambda_0 = 0$ with $n = 0$, the eigenfunctions of (6.1) are

$$\Theta(\theta) = c_1,\ n = 0\text{and } \Theta(\theta) = c_1\ \cos n\theta + c_2\ \sin n\theta,\ n = 1, 2, 3, \ldots \tag{6.7}$$

Now solve (6.1).
For $\lambda_0 = 0$, $n = 0$ Let $R = r^m$, $R' = mr^{m-1}$ and $R'' = m(m-1)r^{m-2}$, substituting into (6.1),

$$\begin{aligned} r^2 m(m-1)r^{m-2} + rmr^{m-1} &= 0 \\ (m(m-1) + m)r^{m-1} &= 0 \\ m &= 0, 0 \end{aligned}$$

So, the solution is

$$R(r) = c_1 + c_2\ \ln\ r \tag{6.8}$$

For $\lambda = n^2$,

$$\begin{aligned} r^2 m(m-1)r^{m-2} + rmr^{m-1} - n^2 r^m &= 0 \\ m^2 - n^2 &= 0 \\ m &= n, -n \end{aligned}$$

The solution is

$$R(r) = c_1 r^n + c_2 r^{-n} \tag{6.9}$$

Apply BC, $u(b, \theta) = R(b)\Theta(\theta) = 0$. Since $\Theta(\theta) \neq 0$, thus $R(b) = 0$ in (6.8) and (6.9) gives

$$R(b) = c_1 + c_2\ \ln\ b = 0 \leftrightarrow c_1 = -c_2\ \ln\ b \tag{6.10}$$

$$So\ R(r) = c_2(\ln\ r - \ln\ b) = c_2\ \ln\ \frac{r}{b} \leftarrow \boxed{\text{from (6.8)}} \tag{6.11}$$

So from (6.7) and (6.11), a product solution when $\lambda_0 = 0$ is

$$u_0(r,\theta) = R(r)\Theta(\theta) \tag{6.12}$$
$$= \left(c_2 \ln \frac{r}{b}\right) c_1 \tag{6.13}$$
$$= A_0 \ln \frac{r}{b} \leftarrow \boxed{c_1 c_2 = A_0} \tag{6.14}$$

Now apply BC; $R(b) = 0$ on (6.9) give

$$R(b) = c_1 b^n + c_2 b^{-n} = 0 \leftrightarrow c_1 = -c_0 b^{-n},\ c_2 = c_0 b^n \tag{6.15}$$
$$So\ R(r) = -c_0 b^{-n} r^n + c_0 b^n r^{-n} \tag{6.16}$$
$$= c_0 \left(\left(\frac{b}{r}\right)^n - \left(\frac{r}{b}\right)^n\right) \tag{6.17}$$

A product solution when $\lambda_n = n^2,\ n = 1, 2, \ldots$ then

$$u_n(r,\theta) = c_0 \left(\left(\frac{b}{r}\right)^n - \left(\frac{r}{b}\right)^n\right)(c_1 \cos n\theta + c_2 \sin n\theta) \tag{6.18}$$

Form (6.14) and (6.18) and by superposition principle then solution is,

$$u(r,\theta) = u_n(r,\theta) + u_n(r,\theta) \tag{6.19}$$
$$= A_0 \ln \frac{r}{b} + \sum_{i=1}^{n} c_0 \left(\left(\frac{b}{r}\right)^n - \left(\frac{r}{b}\right)^n\right)(c_1 \cos n\theta + c_2 \sin n\theta)$$
$$= A_0 \ln \frac{r}{b} + \sum_{i=1}^{n} \left(\left(\frac{b}{r}\right)^n - \left(\frac{r}{b}\right)^n\right)(A_n \cos n\theta + B_n \sin n\theta) \leftarrow \boxed{c_0 c_1 = A_n \text{ and } c_0 c_2 = B_n} \tag{6.20}$$

b) If $a = 1$, $b = 2$ and $f(\theta) = \cos\theta$, determine the specific solution $u(r,\theta)$:
Apply BC $u(a,\theta) = f(\theta)$ gives

$$f(\theta) = u(a,\theta) = A_0 \ln \frac{a}{b} + \sum_{i=1}^{n} \left(\left(\frac{b}{a}\right)^n - \left(\frac{a}{b}\right)^n\right)(A_n \cos n\theta + B_n \sin n\theta) \tag{6.21}$$

(6.22)

where

$$
\begin{aligned}
A_0 \ln \frac{1}{2} &= \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta \\
A_0 &= \frac{1}{2\pi \ln \frac{1}{2}} \int_0^{2\pi} \cos\, \theta\, d\theta \\
&= \frac{1}{2\pi} [\sin\, \theta]_0^{2\pi} \\
&= 0 \\
A_n \left(\left(\frac{b}{a}\right)^n - \left(\frac{a}{b}\right)^n \right) &= \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, \cos\, n\theta\, d\theta \\
A_n &= \frac{1}{\left(\left(\frac{b}{a}\right)^n - \left(\frac{a}{b}\right)^n\right)\pi} \int_0^{2\pi} f(\theta)\, \cos\, n\theta\, d\theta \\
&= \frac{1}{\left(\left(\frac{2}{1}\right)^n - \left(\frac{1}{2}\right)^n\right)\pi} \int_0^{2\pi} \cos\, \theta\, \cos\, n\theta\, d\theta \\
&= \frac{2^1}{(4^1 - 1)\pi} \int_0^{2\pi} \cos^2\, \theta\theta \leftarrow \boxed{n = 1. \text{ If } n \neq 1, \text{ then orthogonal function, } 0} \\
&= \frac{2}{3\pi} \int_0^{2\pi} \frac{1}{2}(1 + \cos\, 2\theta)\, d\theta \\
&= \frac{1}{3\pi} [\theta + \frac{1}{2} \sin\, 2\theta]_0^{2\pi} \\
&= \frac{1}{3\pi}(2\pi) = \frac{2}{3} \\
B_n \left(\left(\frac{2}{1}\right)^n - \left(\frac{1}{2}\right)^n \right) &= \frac{1}{2\pi} \int_0^{2\pi} \cos\, \theta\, \sin\, n\theta\, d\theta \\
B_n &= \frac{1}{\left(\left(\frac{2}{1}\right)^n - \left(\frac{1}{2}\right)^n\right)\pi} \int_0^{2\pi} \cos\, \theta\, \sin\, n\theta\, d\theta \\
&= \frac{2^n}{(4^n - 1)\pi} \int_0^{2\pi} \frac{1}{2}(\sin(1+n)\theta - \sin(1-n)\theta)\, d\theta \\
&= \frac{2^{n-1}}{(4^n - 1)\pi} \left[\frac{1}{1+n}(-\cos(1+n)\theta + \frac{1}{1-n} \cos(1-n)\theta \right]_0^{2\pi} \\
&= \frac{2^{n-1}}{(4^n - 1)\pi} \left[\frac{-1}{1+n}(\cos(1+n)2\pi - 1) + \frac{1}{1-n}(\cos(1-n)2\pi - 1) \right]
\end{aligned}
$$

Consider

$$\begin{aligned}
\cos(1+n)2\pi &= \cos(2\pi)\cos(2n\pi) - \sin(2\pi)\sin(2n\pi) \\
&= 1 \\
\cos(1-n)2\pi &= \cos(2\pi)\cos(2n\pi) + \sin(2\pi)\sin(2n\pi) \\
&= 1
\end{aligned}$$

So

$$B_n = 0$$

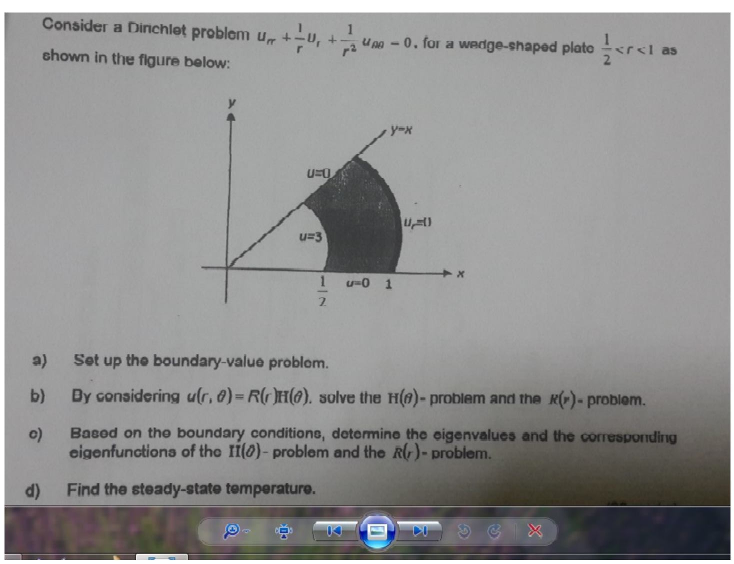
The specific solution

$$u(r,\theta) = \frac{1}{3}\sum_{n=1}^{n}\left(2^n - 2^{-n}\right)\cos n\theta \tag{6.23}$$

6.2 Wedge-shape plate

- The Dirichlet Problem. The problem is to find a harmonic function u inside a domain D so that the values of u are prescribed on the boundary ∂D of D ($u = f$ is given on the boundary D).
- The Neumann Problem. The problem is to find a harmonic function$u$ inside the domain D so that the normal derivatives of u, (i.e. u) are prescribed on the boundary ($\frac{\partial u}{\partial \eta} = g$ on ∂D.) Recall that the normal derivative at a point.

6.2.1 Q5 Jan 2012


Consider a Dirichlet problem $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$, for a wedge-shaped plate $\frac{1}{2} < r < 1$ as shown in the figure below:

a) Set up the boundary-value problem.

b) By considering $u(r,\theta) = R(r)\Theta(\theta)$, solve the $\Theta(\theta)$- problem and the $R(r)$- problem.

c) Based on the boundary conditions, determine the eigenvalues and the corresponding eigenfunctions of the $\Theta(\theta)$- problem and the $R(r)$- problem.

d) Find the steady-state temperature.

Solution

a) BVP

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0,\ \frac{1}{2} < r < 1 \quad (6.24)$$

$$s.t \quad (6.25)$$

$$u(r,0) = 0,\ \frac{1}{2} < r < 1 \quad (6.26)$$

$$u(\frac{1}{2},\theta) = 3,\ 0 < \theta < \frac{\pi}{4} \quad (6.27)$$

$$u_r(1,\theta) = 0,\ 0 < \theta < \frac{\pi}{4} \quad (6.28)$$

$$u(r,\frac{\pi}{4}) = 0,\ \frac{1}{2} < r < 1 \quad (6.29)$$

b) By considering $u(r,\theta) = R(r)H(\theta)$, solve the $H(\theta)$-problem and the $R(r)$-problem. From (6.24),

$$R^{''}H + \frac{1}{r}R^{'}H + \frac{1}{r^2}RH^{''} = 0$$

$$r^2R^{''}H + rR^{'}H + RH^{''} = 0$$

$$r^2R^{''}H + rR^{'}H = -RH^{''}$$

$$\frac{r^2R^{''} + rR^{'}}{R} = -\frac{H^{''}}{H} = \lambda$$

leads to two ODEs

$$r^2R^{''} + rR^{'} - \lambda R = 0 \quad (6.30)$$

$$H^{''} + \lambda H = 0 \quad (6.31)$$

We are looking for

$$H^{''} + \lambda H = 0, \quad (6.32)$$

The three possibles general solution of (6.32),

$$H(\theta) = c_1 + c_2\theta,\ \lambda = 0 \quad (6.33)$$

$$H(\theta) = c_1 \cosh \alpha\theta + c_2 \sinh \alpha\theta,\ \lambda = -\alpha^2 < 0 \quad (6.34)$$

$$H(\theta 0 = c_1 \cos \alpha\theta + c_2 \sin \alpha\theta,\ \lambda = \alpha^2 > 0 \quad (6.35)$$

The solution of Cauchy-Euler (6.30) are as follows. Let $R = r^m, R^{'} = mr^{m-1}, R^{''} =$

$m(m-1)$. Substitute into (6.30) and for $\lambda = 0$, becomes

$$
\begin{aligned}
r^2 m(m-1)r^{m-2} + rmr^{m-1} &= 0 \\
m(m-1)r^m + mr^m &= \\
m^2 r^m &= 0 \\
m &= 0, 0 \leftarrow \boxed{\text{equal real, } R(r) = c_1 r^m + c_2 r^m \ln r}
\end{aligned}
$$

Thus the solution is

$$R(r) = c_1 + c_2 \ln r \tag{6.36}$$

For $\lambda_n = \alpha^2$, (6.30) becomes

$$
\begin{aligned}
r^2 m(m-1)r^{m-2} + rmr^{m-1} - \alpha^2 r^m = 0 \\
(m^2 - \alpha^2) r^m &= 0 \\
m &= \alpha, -\alpha \leftarrow \boxed{\text{real and different olution} R(r) = c_1 r^{m_1} + c_2 r^{-m_2}}
\end{aligned}
$$

Thus the solution is

$$R(r) = c_1 r^{\alpha} + c_2 r^{-\alpha} \tag{6.37}$$

c) Apply BC on H-problem:

The BC (6.26) and (6.29)w5 together with (6.30) constitute a regular Sturm-Liouville problem

Now apply BC (6.26) and (6.29): $u(r,0) = R(r)H(0) = 0$. Since $R(r) \neq 0$, so $H(0) = 0$. $u(r, \pi/4) = 0 \rightarrow H(\frac{\pi}{4}) = 0$.
The regular Sturm-Liouville problem:

$$H'' + \lambda H = 0, \; H(0) = 0, \; H(\frac{\pi}{4}) = 0. \tag{6.38}$$

From (6.33) gives

$$H(0) = c_1 + c_2(0) = 0 \rightarrow c_2 = 0$$

$$So \; H(\theta) = c_1 \tag{6.39}$$

$$H(\frac{\pi}{4}) = c_1 = 0 \rightarrow c_1 = 0 \tag{6.40}$$

Thus $u(r,\theta) = 0$ when $\lambda = 0$.
From (6.34),

$$
\begin{aligned}
u(r,0) &= c_1 \cosh(0) + c_1 \sinh(0) = 0 \\
&= c_1 = 0 \rightarrow \; c_1 = 0.
\end{aligned}
$$

$$So \, u(r,\theta) = c_2 \sinh \alpha\theta \tag{6.41}$$

From BC (6.29): $u(r, \frac{\pi}{4}) = R(r)H(\frac{\pi}{4}) = 0 \rightarrow H(\frac{\pi}{4}) = 0.$

$$u(r, \frac{\pi}{4}) = c_1 \sinh \alpha\frac{\pi}{4}$$

This solution is not unbounded and nonperiodic unless $c_1 = 0$, So the solution is trivial.
Now apply BC (6.26) and (6.29) on (6.35),

$$\begin{aligned} u(r,0) &= c_1 \cos(0) + c_2 \sin(0) = 0 \\ &= c_1 = 0 \rightarrow c_1 = 0 \end{aligned}$$

For nontrivial solution $c_2 \neq 0$, so $\sin \alpha\frac{\pi}{4} = 0 \rightarrow \alpha = 4n$. So the problem (6.38) possesses eigenvalues $\lambda_n = 4n,\ n = 1, 2, \ldots$. The eigenfunction is

$$H(\theta) = c_2 \sin 4n\theta,\ \ n = 1, 2, 3, \ldots \tag{6.42}$$

Apply BC on R-problem:
Transform BC (6.27) give $R'(1) = 0$. From (6.36)

$$\begin{aligned} R(r) &= c_1 + c_2 \ln r \\ R'(r) &= \frac{c_2}{r} \\ R'(0) &= \frac{c_2}{1} \rightarrow c_2 = 0 \end{aligned}$$

So the solution is trivial.
Since we want $R(r)$ to be bounded as $r \rightarrow 0$, so eqn (6.37) we find that $c_2 = 0$ and $\lambda = \alpha = 4n$. Thus (6.37) becomes

$$R(r) = c_1 r^{4m} \tag{6.43}$$

d) From (6.42) and (6.43) we obtain a product solution

$$\begin{aligned} u_n(r,\theta) &= R(r)H(\theta) \\ &= c_1 r^{4n}(c_2 \sin 4n\theta) \\ &= A_n r^{4n} \sin 4n\theta \end{aligned}$$

And by using the superposition principle we obtain the solution of the steady-state temperature,

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{4n} \sin 4n\theta \tag{6.44}$$

Now find A_n: apply BC $u(1/2, \theta) = 3$

$$
\begin{aligned}
u(\frac{1}{2}, \theta) = 3 &= \sum_{n=1}^{\infty} A_n \left(\frac{1}{2}\right)^{4n} \\
A_n \left(\frac{1}{2}\right)^{4n} &= \int_0^{\frac{\pi}{4}} 3 \sin 4n\theta \, d\theta \\
&= \frac{3}{4n}[-\cos 4n\theta]_0^{\frac{\pi}{4}} \\
&= \frac{3}{4n}(-\cos 4n\frac{\pi}{4} + 1) \\
&= \frac{3}{4n}(1 - (-1)^n) \\
A_n &= \frac{3 \cdot 2^{4n}}{4n}(1 - (-1)^n)
\end{aligned}
$$

Therefore the steady-state temperature is

$$
u(r, \theta) = \frac{3}{4} \sum_{n=1}^{\infty} \frac{(2r)^{4n}(1 - (-1)^n)}{n} \sin 4n\theta \tag{6.45}
$$