Finite Element Method by Example in Qt/C++

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Abstract

Learn FEM by example in a few steps.

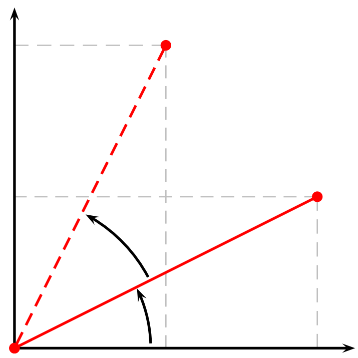
1 Introduction

I spent some time lerning FEM using the Internet resources only. I could not find a complete explanation in one place so I decided to create one by putting togather all information I found. Below work is an output of self education. Just curious where I will get from here! All suggestions and comments are welcome!

2 Rotation of Objects

2.1 Rotation of Point 2D

given from geometry



$$x_0 = r\cos\alpha$$
$$y_0 = r\sin\alpha$$
$$x_1 = r\cos(\alpha+\beta) = r\cos\alpha\cos\beta - r\sin\alpha\sin\beta$$
$$y_1 = r\sin(\alpha+\beta) = r\sin\alpha\cos\beta + r\cos\alpha\sin\beta$$

hence

$$x_1 = x_0\cos\beta - y_0\sin\beta$$
$$y_1 = x_0\sin\beta + y_0\cos\beta$$

2.2 Rotation Matrix 3D

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

2.3 Example of Rotated Cube

2.4 Element Stiffness Matrix in Local Coordinates

Relation between axial forces, q1, q2, and axial displacements, u1, u2, only (in local coordinates).

$$\mathbf{k} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{q} = \mathbf{k}\ \mathbf{u}$$

2.5 Coordinate Transformation

Global and local coordinates

$$L = \sqrt{(x_b - x_a)^2 + (y_b - y_a)^2 + (z_b - z_a)^2}$$
$$\cos x = \frac{x_b - x_a}{L} = c_x$$
$$\cos y = \frac{y_b - y_a}{L} = c_y$$
$$\cos z = \frac{z_b - z_a}{L} = c_z$$

Displacements

$$u_a = a_x \cos x + a_y \cos y + a_z \cos z$$
$$u_b = b_x \cos x + b_y \cos y + b_z \cos z$$

$$\begin{bmatrix} u_a \\ u_b \end{bmatrix} = \begin{bmatrix} c_x & c_y & c_z & 0 & 0 & 0 \\ 0 & 0 & 0 & c_x & c_y & c_z \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \\ b_x \\ b_y \\ b_z \end{bmatrix}$$

$$\mathbf{u} = \mathbf{T}\ \mathbf{v}$$

Forces

$$\begin{bmatrix} f_{ax} \\ f_{ay} \\ f_{az} \\ f_{bx} \\ f_{by} \\ f_{bz} \end{bmatrix} = \begin{bmatrix} c_x & 0 \\ c_y & 0 \\ c_z & 0 \\ 0 & c_x \\ 0 & c_y \\ 0 & c_z \end{bmatrix} \begin{bmatrix} q1 \\ q2 \end{bmatrix}$$

$$f = T^T q$$

2.6 Element Stiffness Matrix in Global Coordinates

$$\begin{bmatrix} q1 \\ q2 \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u1 \\ u2 \end{bmatrix}$$

$$f = TTqu = Tv$$

$$q = ku$$

$$q = kTv$$

$$TTq = TTkTv$$

$$f = TTkTv$$

$$f = Kv$$

$$K = \frac{EA}{L} \begin{bmatrix} c_x^2 & c_xc_y & c_xc_z & -c_x^2 & -c_xc_y & -c_xc_z \\ c_xc_y & c_y^2 & c_yc_z & -c_xc_y & -c_y^2 & -c_yc_z \\ c_xc_z & c_yc_z & c_z^2 & -c_xc_z & -c_yc_z & -c_z^2 \\ -c_x^2 & -c_xc_y & -c_xc_z & c_x^2 & c_xc_y & c_xc_z \\ -c_xc_y & -c_y^2 & -c_yc_z & c_xc_y & c_y^2 & c_yc_z \\ -c_xc_z & -c_yc_z & -c_z^2 & c_xc_z & c_yc_z & c_z^2 \end{bmatrix}$$

2.7 Stiffness Matrix Construction

Consider linear equations with exactly one solution:

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Fixing y to 0 one of constants must be freed (c_2 has been chosen). Moving variables to left and constants to right side we get

$$a_1x = c_1$$

$$a_2x - c_2 = 0$$

$$\begin{bmatrix} a_1 & 0 \\ a_2 & -1 \end{bmatrix} \begin{bmatrix} x \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$$